Al-Nahrain Journal of Science
ANJS, Vol.25 (3), September, 2022, pp. 20-24

# An Efficient Approach to Approximate the Solutions of Fractional Partial Integro-Differential Equations

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**Article's Information**

Received:
30.05.2022
Accepted:
16.06.2022
Published:
30.09.2022

**Keywords:**
Partial integro-differential equations
Conformable fractional derivative
Legendre polynomials
Semi-analytic method

**Abstract**

In this article a computational efficient approach is presented so as to examine the approximate solutions (AP) of fractional One-dimensional partial integro-differential equations (CF1DPDEs). The fractional derivative will be in the conformable sense. The suggested approach combined between the shifted Legendre polynomials and a semi-analytic approach. the proposed approach is tested by some examples which are introduced to illustrate its accuracy, applicability and efficiency.

DOI: 10.22401/ANJS.25.3.04
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## 1. Introduction

Recently, the subject of fractional calculus is considered as a powerful tool and has been gaining considerable attention of different scientist due its important role in physics and engineering processes. Fractional derivatives, may be treated as a popularization of integer derivatives and have been greatly used in characterizing phenomenon in physics and biology see for instance [8,10]. There are many interesting kinds for the fractional order derivatives such as Riemann-Liouville, Grunwald-Letnikov and Caputo, etc. [8,10]. A few years ago, Khalil et. al [7] was first introduced another kind of fractional derivative which is so-called conformable fractional derivative (CFD) which fixed the problems that have been occurred in the other types of the fractional derivatives mentioned above which is the inheritance of the nonlocal properties from integral. (CFD) is a domestic fractional derivative that simulate the traditional derivative [1].

This paper fastens on finding the solutions of the conformable fractional One-dimensional partial integro-differential equations. Partial integro-differential equations of fractional order are a very extensive field which has a numerous application in different fields of science and engineering such as mathematical biology, population dynamics, chemical processes, hydraulic system and communication networks [4]. Partial integro-differential equations of fractional order have been attacked by many researchers such as [2,3,6,9,11].

## 2. Preliminaries

In this section, some of the most important concepts related to this paper will be presented for completeness purpose.

**Definition 2.1, [1,7].** Given a function $f:[0,\infty)\to\mathbb{R}$, then the conformable fractional derivative (CFD) of order $\alpha$ of $f$ is given by:

$$CD_t^{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} \qquad \ldots(1)$$

for all $t>0$, $\alpha\in(0,1]$.

**Definition 2.2, [1,7].** The conformable integral of order α is defined by:

$$CI_t^{\alpha}(f)(t)=\int_a^t f(x)x^{\alpha-1}dx,\ a\geq 0 \qquad \ldots(2)$$

**Theorem 2.1, [7].** Let α ∈ (0, 1] and h, k be α-differentiable at a point t > 0, then

1. $CD_t^{\alpha}(ah+bk)=a(CD_t^{\alpha}h)+b(CD_t^{\alpha}k), \forall a,b\in\mathbb{R}$ …(3)
2. $CD_t^{\alpha}(t^p)=pt^{t-\alpha}, \forall p\in\mathbb{R}$ …(4)
3. $CD_t^{\alpha}(\lambda)=0$, for all constant function $h(t)=\lambda$ …(5)
4. $CD_t^{\alpha}(hk)=h(CD_t^{\alpha}k)+k(CD_t^{\alpha}h)$ …(6)
5. $CD_t^{\alpha}(h/k)=k(CD_t^{\alpha}h)-h(CD_t^{\alpha}k)/k^2$ …(7)
6. If, in addition, his differentiable, then:

$$(CD_t^{\alpha}h)(t)=t^{1-\alpha}\frac{dh}{dt}(t) \qquad \ldots(8)$$

## 3. Shifted Legendre Polynomials (SHLPs)

The Legendre polynomial of degree $i$ denoted by $P_i(v)$, defined on the interval $[-1,1]$ can be generated as follows [5]:

$$P_{i+1}(v)=\frac{(2i+1)}{(i+1)}vP_i(v)-\frac{i}{i+1}P_{i-1}(v), i=1,2,\ldots$$

On the other hand, the (SHLPs) denoted by $P_{l,i}(t)$ defined on [0,1] can be introduced if we suppose that $v=\frac{2t}{l}-1$, then the polynomials are constructed using the following formula for all $i=1,2,\ldots$:

$$P_{l,i+1}(t)=\frac{(2i+1)}{(i+1)}\left(\frac{2t}{l}-1\right)P_{l,i}(t)-\frac{i}{i+1}P_{l,i-1}(t) \quad \ldots(9)$$

$$P_{l,0}(t)=1 \text{ and } P_{l,1}(t)=\frac{2t}{l}-1,\ l\neq 0$$

The orthogonality relation can be presented as:

$$\int_0^1 P_{l,i}(t)P_{l,j}(t)dt=\begin{cases}\frac{i}{2i+1}, & i=j\\ 0, & i\neq j\end{cases}$$

A function $y(t)\in P_2(0,1)$ can be decomposed in terms of the (SHLPs) as:

$$y(t)=\sum_{i=0}^{\infty}a_i P_{l,i}(t) \quad \ldots(10)$$

where the coefficients $a_i$ are given by:

$$a_i=\frac{(2i+1)}{l}\int_0^l y(t)P_{l,i}(t)dt, i=0,1,\ldots \quad \ldots(11)$$

## 4. The Approach

In this section the proposed technique will be introduced in order to find the approximate solution of the following problem:

$$CD_t^{\alpha}u(x,t)=g(x,t)+CI_x^{\beta}k(x,t)F\left[u(x,t)\right] \quad \ldots(12)$$

subject to:

$$u(x,0)=\varphi(x),\ x\in[a,b] \quad \ldots(13)$$

where $g(x,t)$ is a continuous function. To start the procedure, first note that (12) can be written as:

$$\chi\left(u(x,t)\right)+N(x,t)+f(x,t)=0 \quad \ldots(14)$$

where:

$$\chi\left(u(x,t)\right)=CD_t^{\alpha}u(x,t), N\left(u(x,t)\right)=-CI_x^{\beta}k(x,t)F\left[u(x,t)\right]$$

and $f(x,t)=-g(x,t)$.

Now we follow the next steps:

**Step 1:** We assume that $u_0(x,t)$ is an initial guess solution to the problem (12)-(13) and it satisfy the following equation:

$$\chi\left(u_0(x,t)\right)+f(x,t)=0 \quad \ldots(15)$$

subject to:

$$u_0(x,0)=\varphi(x) \quad \ldots(16)$$

Therefore, $u_0(x,t)=\varphi(x)+CI_t^{\alpha}g(x,t)$, the initial guess $u_0(x,t)$ can be decomposed in terms of the (SHLPs) as:

$$u_0^*(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}P_{l,i}(x)P_{l,j}(t) \quad \ldots(17)$$

where $m$ and $n$ are some fixed constants and for all $i=0,\ldots,n,\ j=0,\ldots,m$

$$a_{ij}=(2i+1)(2j+1)\int_0^1\int_0^1 u_0(x,t)P_{l,i}(x)P_{l,j}(t)dx\,dt \quad \ldots(18)$$

**Step 2:** To find the next iteration, we solve the following problem:

$$\chi\left(u_1(x,t)\right)+f(x,t)+N\left(u_0^*(x,t)\right)=0 \quad \ldots(19)$$

subject to:

$$u_1(x,0)=\varphi(x) \quad \ldots(20)$$

Hence

$$u_1(x,t)=\varphi(x)+CI_t^{\alpha}g(x,t)+CI_t^{\alpha}CI_x^{\beta}k(x,t)F\left[u_0^*(x,t)\right] \quad \ldots(21)$$

similarly, $u_1(x,t)$ can be decomposed in terms of the (SHLPs) as:

$$u_1^*(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}P_{l,i}(x)P_{l,j}(t) \quad \ldots(22)$$

where:

$$a_{ij}=(2i+1)(2j+1)\int_0^1\int_0^1 u_1(x,t)P_{l,i}(x)P_{l,j}(t)dx\,dt \quad \ldots(23)$$

for all , $i=0,\ldots,n,\ j=0,\ldots,m$

**Step 3:** For finding $u_2(x,t),u_3(x,t),\ldots$ etc. an iterative procedure can be given as:

$$\chi\left(u_{k+1}(x,t)\right)+f(x,t)+N\left(u_k^*(x,t)\right)=0,\ k=1,2,3,\ldots. \quad \ldots(24)$$

subject to:

$$u_{k+1}(x,0)=\varphi(x) \quad \ldots(25)$$

Hence:

$$u_{k+1}(x,t)=\varphi(x)+CI_t^{\alpha}g(x,t)+CI_t^{\alpha}CI_x^{\beta}k(x,t)F\left[u_k^*(x,t)\right] \quad \ldots(26)$$

Similarly, $u_{k+1}(x,t)$ will be decomposed in terms of the (SHLPs) as:

$$u_{k+1}^*(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}P_{l,i}(x)P_{l,j}(t) \quad \ldots(27)$$

where:

$$a_{ij}=(2i+1)(2j+1)\int_0^1\int_0^1 u_{k+1}(x,t)P_{l,i}(x)P_{l,j}(t)dx\,dt \quad \ldots(28)$$

for all $i=0,\ldots,n,\ j=0,\ldots,m$.

Note that each of the $u_k(x,t), k=0,1,2,\ldots$ is a solution to the problem (12)-(13).

## 5. Illustrative Examples

In order to demonstrates the accuracy and efficiency of the proposed method several applications are given below:

**Example 5.1.** Examine the following nonlinear (CF1DPDEs)

$$CD_t^{\frac{3}{4}} ux,t = g(x,t) + \int_0^t (x-s)[u(x,s)]^2 ds \qquad \ldots(29)$$

Subject to:

$$u_0(x,t) = 0 \qquad \ldots(30)$$

where $g(x,t) = \frac{t^3 x}{3} - \frac{t^2 x^2}{2} + t^{\frac{1}{4}} x$ and the exact solution is $u(x,t) = xt$.

Following Figures 1-3 represents a comparison between the exact solution of problem (29)-(30) and the (AP) obtained by using the proposed approach at different values of $t$.

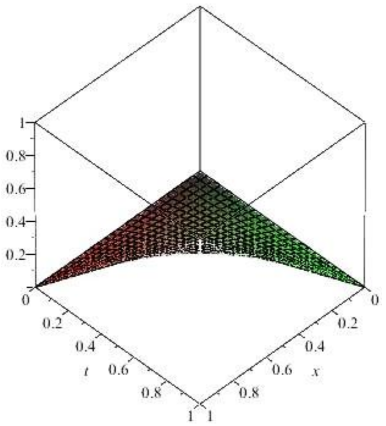


**Figure 1.** The approximate solution of problem (29)-(30).

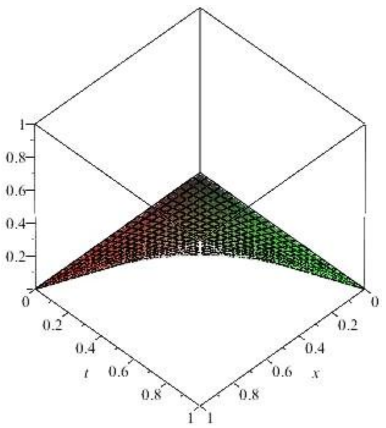


**Figure 2.** The exact solution of problem (29)-(30).

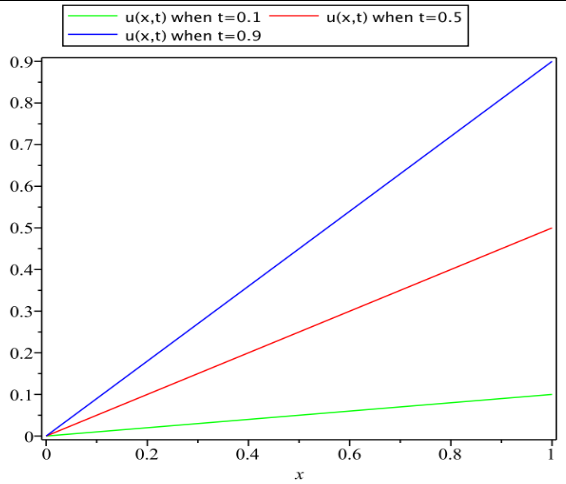


**Figure 3.** The approximate solutions of problem (29)-(30) at different values of $t$.

**Example 5.2.** Given the following linear (CF1DPDEs)

$$CD_t^{0.6} u(x,t) = g(x,t) + CI_x^{0.6}(x-t)u(x,t) \qquad \ldots(31)$$

Subject to:

$$u_0(x,t) = 0 \qquad \ldots(32)$$

where $g(x,t) = 0.625t^3 x^{\frac{8}{5}} - 0.3846t^2 x^{\frac{13}{5}} + 2t^{1.4} x$ and the exact solution is $u(x,t) = xt^2$. Following Figures 4-6 represents a comparison between the exact solution of problem (31)-(32) and the (AP) obtained by using the proposed approach at different values of $t$.

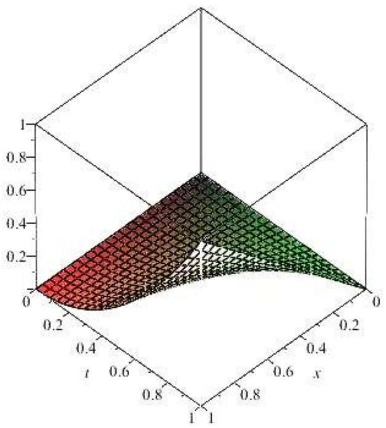


**Figure 4.** The approximate solution of problem (31)-(32).

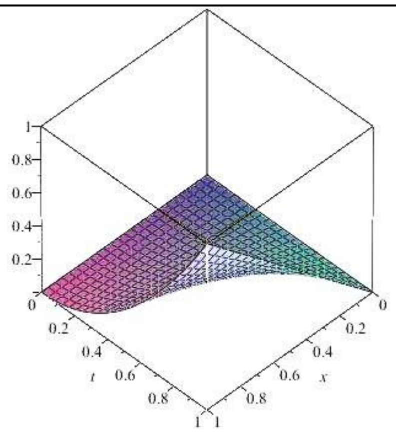


**Figure 5.** The exact solution of problem (31)-(32).

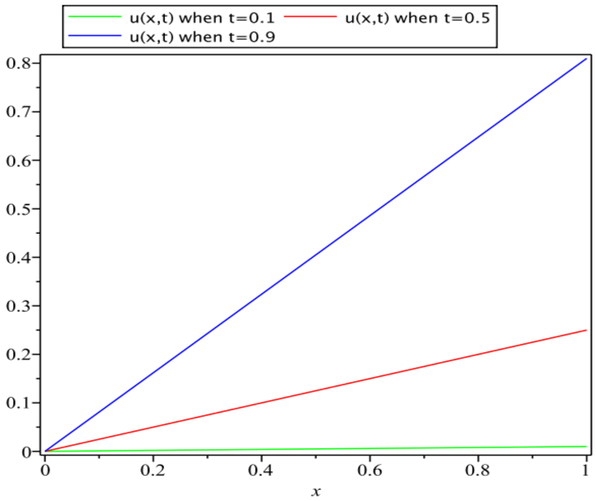


**Figure 6.** The approximate solutions of problem (31)-(32) at different values of *t*.

**Example 5.3.** Examine the nonlinear (CF1DPDEs) given below:

$$CD_t^{0.6}u(x,t)=g(x,t)+CI_x^{0.6}(x-t)[u(x,t)]^2 \quad \ldots(33)$$

Subject to:

$$u_0(x,t)=0 \quad \ldots(34)$$

where $g(x,t)=t^{0.4}x-0.27778t^2x^{\frac{18}{5}}+0.38461t^3x^{\frac{13}{5}}$ and the exact solution is $u(x,t)=xt$ . Following Figure 7 represents the (AP) of problem (33)-(34)

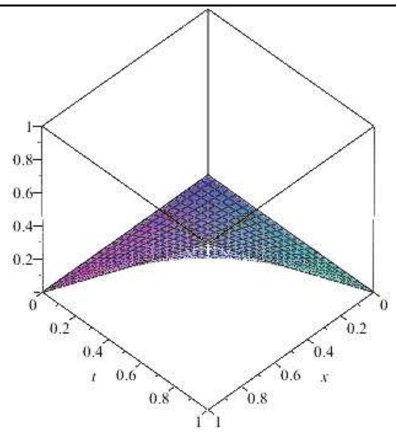


**Figure 7.** The approximate solution of problem (33)-(34).

## 6. Conclusions

A hybrid approach was performed in this article for the sake of finding the (AP) of the (CF1DPDEs). The proposed approach is very simple to apply and its performance is effective and reliable in solving both linear and nonlinear problems as shown in the given examples.

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