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Best One-Sided Multiplier Approximation by Operators

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Abstract

The aim of studying this research is to find the best one-sided multiplier approximation of unbounded function in $L_{p,\psi_n}(X)-$space, $X=[0,1]$, $p\geq 1$ by using type of operators $\mathbb{g}_n(f), \mathbb{G}_n(f)$ by means of operators of algebraic polynomials as well as to show the relationship between the multiplier averaged modules of smoothness (τ-modules) and variation of unbounded functions in $L_{p,\psi_n}(X)$ $-$space.

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1. Introduction

The approximation theory is a theory concerned of how complex functions are approximated by simpler functions. Many researchers in the field of approximation theory especially the approximation of one-sided function. In this paper we study the one-sided multiplier approximation of unbounded function

Among many researchers who have worked in the study of one-sided approximation are; In 2008 [1], Oleksanor studied one-sided weighted approximation by polynomials on the real line in L_P-space and obtained some results and studied one-sided weighted approximation by polynomials on the real line in L_P-space and obtained results. In 2010 [2], Motornyi and Pas'ko studied the best one-sided approximation through a class of some differentiable function in L_1-space. In 2012 [3] Rensuoli and Yong, have studied the best m-term one-sided approximation by the trigonometric polynomials on some classes of Besov space in $L_p(T_d)$, $p \geq 1$. In 2017 [4]. Huseyin and Ramazan studied one-sided approximation with averaged modulus of smoothness of function $f \in L_\varphi$ -space by trigonometric polyno midls and got direct and inverse theorems. In 2020 [5], Saheb et al. studied the best multiplier approximation of unbounded periodic functions in $L_{p,\psi_n}(B)$, $B=[0,2\pi]$, where they used discrete operator positiue, ard there were results.

2. Basic Definitions

In this research, we will need some definitions, which are given next:

Definition 2.1, [8]. A series $\sum_{n=0}^{\infty} a_n$ is called a multiplier convergent if there is a sequence $\{\psi_n\}_{n=0}^{\infty}$, such that $\sum_{n=0}^{\infty} a_n \psi_n < \infty$ and we will say that $\{\psi_n\}_{n=0}^{\infty}$ is a multiplier for the convergence.

Definition 2.2. For any real valued function f defined on $X=[a,b]$ if there is a sequence $\{\psi_n\}_{n=0}^{\infty}$, such that $\int_X f\,\psi_n(x)\,dx < \infty$, then we say that ψ_n is the multiplier integral.

Definition 2.3. Let $f \in L_{p,\psi_n}(X)$, $X=[0,1]=[a,b]$, $p \in [1,\infty)$, then the space of all real valued unbounded functions f, such that $\int_X f\,\psi_n(x)\,dx < \infty$ is defined by:

$$\| f \|_{p,\psi_n} = \left(\int_X | f(x)\psi_n|^p\,dx\right)^{1/p}, x \in X$$

Definition 2.4. For $f \in L_{p,\psi_n}(X)$, $X=[a,b]$, $0<\delta$, we will define the following concepts:

$$\omega(f,\delta)_{p,\psi_n} = \underbrace{sup}_{|h|<\delta} \|f(x+h)-f(x)\|_{p,\psi_n}$$

is the multiplier integral modulus of function f.

The multiplier local of smoothness for f of order k at point $x \in [a,b], \delta \in \left[0,\frac{b-a}{k}\right]$ is defined by:

$$\omega_k(f,x,\delta) = \sup_{|h|<\delta}\{|\Delta_h^k f\psi_n(t)| : t, t+kh \in [x-\tfrac{k\delta}{2}, x+\tfrac{k\delta}{2}] \cap [a,b]\}.$$

where the difference of function f is:

$\Delta_h^k f(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x - \frac{k\delta}{2} + ih),\ x \mp \frac{kh}{2} \in X$

$\tau_k\ (f,\delta)_{p,\psi_n} = \|\omega_k\ (f,.;\delta)\|_{p,\psi_n},\ p \in [1,\infty),\ k \in \mathrm{N}$

is the multiplier averaged modulus of smoothness of f of order k.

Definition 2.5, [7]. The degree of best one-sided approximation of f is:

$\tilde{E}_n(f)p = \inf\{\|\ p_n - q_n\|_{L\,p\,(X)} : q_n(x) \le f(x) \le p_n(x)\}$

Also, the degree of best approximation of a function $f \in Lp(X)$ is define by:

$E_n(f)\,p = \inf\{\|\ f - p_n\|_{L\,p\,(X)} : p_n \in P_n\}$

Definition 2.6. The degree of best one-sided multiplier approximation of f is:

$\tilde{E}_n(f)\,p,\psi_n = \inf\{\|\ p_n - q_n\|_{L\,p,\psi_n\,(X)} : q_n(x) \le f(x) \le p_n\,(x)\}$

Also, the degree of best multiplier approximation of a function $f \in Lp,\psi_n(X)$ is define by:

$E_n(f)p,\psi_n = \inf\{\|\ f - p_n\|_{L\,p,\psi_n\,(X)} : p_n \in P_n\}$

3. Auxiliary Lemmas

In this section some results obtained throughout this work are presented, which are termed as an auxiliary lemmas.

Lemma 3.1, [6]. Let $n \in N, t \in [0,1]$, then the algebraic polynomials $h_n(x,t)$ and $H_n(x,t)$ are called Hermite interpolation polynumials of maximal deyree $2n$ in x satisfy the following:

(i) $h_n(x,t) \le G(x,t) \le H_n(x,t)$.

(ii) $[H_n(x,t) - h_n(x,t)] = (1-(x-1)^2)^n, x \in [0,1]$,

where:

$$G(x,t) = \begin{cases} 0 & \text{if } x < t \\ 1 & \text{if } x \geqslant t \end{cases}$$

Definition 3.1. Let $f \in L_{p,\psi_n}(X), X = [0,1]$, $n \in N$, we define:

$$(f\psi_n)^+(x) = \frac{1}{2}\Big(V_{f\psi_n}(x) + f\psi_n(x)\Big)$$

$$(f\psi_n)^-(x) = \frac{1}{2}\Big(V_{f\psi_n}(x) - f\psi_n(x)\Big)$$

where $(f\psi_n)^+$ and $(f\psi_n)^-$ are non-decreasing functions and $V_{f\psi_n}(x)$ is the multiplier total variation of f on $[0, x]$, and

$$\mathbb{g}_n(f) = f(x) + \int_0^1 h_n(x,t)\ d(f\psi_n)^+(t) - \int_0^1 H_n(x,t)\ d(f\psi_n)(t)$$

$$\mathbb{G}_n(f) = f(x) + \int_0^1 H_n(x,t)\ d(f\psi_n)^+(t) - \int_0^1 h_n(x,t)\ d(f\psi_n)^-(t)$$

Lemma 3.2. Let $f \in L_{p,\psi_n}(X), X = [0,1]$, then:

$\tau(f,n\delta)_{p,\psi_n} \le n\tau(f,\delta)_{\,p,\psi_n},\ \delta > 0, n \in N, p \ge 1$

Proof. Let $[0,1] = [a,b]$,

$$\omega_k(f,x,\delta) = \sup\left\{\left|\Delta_h^k f(t)\psi_n\right| : t, t+kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right] \cap [a,b]\right\}$$

$$\omega_1(f,x;n\delta) = \sup\left\{|f(t+h)\psi_n - f(t)\psi_n| : t, t+h \in \left[x - \frac{n\delta}{2}, x + \frac{n\delta}{2}\right] \cap [a,b]\right\}$$

Let us define:

$\mu_i(x) = x - (n-2i+1)\frac{\delta}{2}, i = 1,2,\ldots,n$

clear that $\mu_i(x) < x, x-(n-2i+1) < x$, we have:

$$\omega_1(f,x;n\delta) \le \sum_{i=1}^{n} \omega_1(f,\mu_i(x),\delta)_{p,\psi_n}$$

$$\left(\int_X |\ \omega_1(f\psi_n,x;n\delta)|^p\,dx\right)^{1/p} \le \sum_{i=1}^{n}\left(\int_X |\ \omega_1(f,\mu_i(x),\delta)|^p\,dx\right)^{1/p}$$

$$\tau(f,n\delta)_{p,\psi_n} \le \sum_{i=1}^{n}\left\{\int_a^b \left(\omega_1\left(f, x - \frac{n-2i+1}{2}\delta;\delta\ \right)\right)^p dx\right\}^{1/p} \le \sum_{i=1}^{n}\tau(f,\delta)_{p,\psi_n}$$

$$\tau(f,n\delta)_{p,\psi_n} \le n\tau(f,\delta)_{p_1\psi_n} \quad \blacksquare$$

Lemma 3.3. Let $f \in L_{p,\psi_n}(X), X = [0,1]$, $1 \le \mathrm{p} < \infty, \delta > 0$, then:

$\tau_1(f,\delta)_{p,\psi_n} \le \delta \bigvee_{[a,b]} f\psi_n$

Proof. Let $X = [0,1] = [a,\mathrm{b}]$ and $f(x) = f(a),\ \forall\, x < a$, $f(x) = f(b), \forall\, x > b$

$$\omega_1(f,x,\delta) = \sup\left\{|\Delta_h^1 f(t)\psi_n| : t, t+h \in \left[x - \frac{\delta}{2}, x + \frac{\delta}{2}\right]\right\} = \sup\{|f(t+h)\psi_n - f(t)\psi_n|\} = \sup\{\mathrm{V}_{[t,t+h]} f\psi_n\} \le \mathrm{V}_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} f\psi_n$$

$$\left(\int_X |\ \omega_1(f\psi_n,x;\delta)|^p\,dx\right)^{1/p} \le \left(\int_a^b \left|\mathrm{V}_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} f\psi_n\ \right|^p dx\right)^{1/p}$$

$$\tau_1(f,\delta)_{p,\psi_n} \le \left(\int_a^b \left|\mathrm{V}_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} f\psi_n\ \right|^p dx\right)^{1/p}$$

$$= \int_a^b V_{\mathrm{x}-\frac{\delta}{2}}^{\mathrm{x}+\frac{\delta}{2}} f(x)\psi_n dx$$

$$= \int_a^b V_{a-\frac{\delta}{2}}^{\mathrm{x}+\frac{\delta}{2}} f(x)\psi_n dx - \int_a^b V_{\alpha-\frac{\delta}{2}}^{x-\frac{\delta}{2}} f(x)\psi_n dx$$

$$= \int_{a+\frac{\delta}{2}}^{b+\frac{\delta}{2}} V_{a-\frac{\delta}{2}}^{t} f(t)\psi_n dt - \int_{a-\frac{\delta}{2}}^{b-\frac{\delta}{2}} V_{a-\frac{\delta}{2}}^{t} f(t)\psi_n dt$$

$$\le \int_{b-\frac{\delta}{2}}^{b+\frac{\delta}{2}} V_{a-\delta/2}^{t} f(t)\psi_n dt$$

$$\le \int_{b-\frac{\delta}{2}}^{b+\frac{\delta}{2}} V_a^b f(t)\psi_n dt$$

$$= \delta V_a^b f\psi_n \quad \blacksquare$$

Lemma 3.4. Let $f \in L_{p,\psi_n}(X), X = [0,1], \delta > 0$, then:

$\omega_k(f,\delta)_{p,\psi_n} \le \delta V_a^b f\psi_n$

Proof.

$$\omega_k(f,\delta)_{p,\psi_n} = \underbrace{sup}_{o\leq h\leq\delta} \left\{\int_a^{b-kh} \left|\Delta_h^k f(x)\psi_n\right|^p dx\right\}^{1/p}$$

$$\begin{aligned}\omega_1(f,\delta)_{p,\psi_n} &= \int_a^{b-h} |\Delta_h f(x)\psi_n| dx \\ &= \int_a^{b-h} |[f(x+h)-f(x)]\psi_n| dx \\ &\leq \int_a^{b-h} \mathrm{V}_{[x+h,x]} f(x)\psi_n dx \\ &= \int_a^{b-h} [f(x+h)\psi_n - f(x)\psi_n] dx \\ &= \int_a^{b-h} [f(x+h)\psi_n - f\psi_n(a) - f(x)\psi_n + f\psi_n(a)] dx \\ &= \int_a^{b-h} \{f(x+h)\psi_n - f\psi_n(a) - [f(x)\psi_n - f\psi_n(a)]\} dx \\ &= \int_a^{b-h} V_a^{x+b} f\psi_n - V_a^x f\psi_n dx \\ &= \int_a^{b-h} V_a^{x+h} f(x)\psi_n dx - \int_a^{b-h} V_a^x dx \\ &= \int_{a+h}^{b} V_a^x f(x)\psi_n dx - \int_a^{b-h} V_a^x dx \\ &= \int_{b-h}^{b} V_a^x f(x)\psi_n dx - \int_a^{a+h} V_a^x f(x)\psi_n dx \\ &\leq \int_{b-h}^{b} V_a^x f(x)\psi_n dx \\ &\leq hV_a^b f\psi_n \\ &\leq \delta V_a^b f\psi_n, \text{since} |h| \leq \delta. \quad \blacksquare\end{aligned}$$

Lemma 3.5. Let $f \in L_{p,\psi_n}(X), X = [0,1], n \in N$, then:
(a) $\mathbb{g}_n(f), \mathbb{G}_n(f\psi_n) \in \pi_{2n}$.
(b) $\mathbb{g}_n(f) \leq f(x) \leq \mathbb{G}_n(f)$.
(c) $\mathbb{G}_n(f) - \mathbb{g}_n(f) = \int_0^1 (1-(x-t)^2)^n dV_{f\psi_n}(t)$.

Proof. (a) Since $h_n(x,t)$ and $H_n(x,t)$ are algebraic polynomials, then $\mathbb{g}_n(f)$ and $\mathbb{G}_n(f) \in \prod_{2n}$.
(b)
$$\begin{aligned} f(x) &= f(x)+f(x)-f(x) \\ &= f(x) + f^+(x) - f^-(x) - [f^+(x) - f^-(x)] \\ &= f(x) + \int_o^x d(f\psi_n)^+(t) - \int_0^x d(f\psi_n)^-(\mathrm{t}) \\ &\geq f(x) + \int_0^x h_n(x,t) d(f\psi_n)^+(t) - \int_0^x H_n(x_1 t) d(f\psi_n)^-(\mathrm{t}) \\ &\geq f(x) + \int_0^1 h_n(x,t) d(f\psi_n)^+(t) - \int_0^1 H_n(x,t) d(f\psi_n)^-(t) \\ &= \mathbb{g}_n(f)\end{aligned}$$

Hence:

$$f(x) \geq \mathbb{g}_n(f) \qquad \ldots(1)$$

Now:

$$\begin{aligned} f(x) &= f(x)+f(x)-f(x) \\ &= f(x) + f^+(x) - f^-(x) - [f^+(x) - f^-(x)] \\ &= f(x) + \int_o^x d(f\psi_n)^+(t) - \int_0^x d(f\psi_n)^-(t) \\ &\leq f(x) + \int_0^x H_n(x,t) d(f\psi_n)^+(t) - \int_0^x h_n(x_1 t) d(f\psi_n)^-(\mathrm{t}) \\ &\leq f(x) + \int_0^1 H_n(x,t) d(f\psi_n)^+(t) - \int_0^1 h_n(x,t) d(f\psi_n)^-(t) \\ &= \mathbb{G}_n(f)\end{aligned}$$

Hence:

$$f(x) \leq \mathbb{G}_n(f) \qquad \ldots(2)$$

from (1) and (2), we get $\mathbb{g}_n(f) \leq f(x) \leq \mathbb{G}_n(f)$
(c) from definition, we have:

$$\mathbb{g}(f) = f(x) + \int_0^1 h_n(x,t) d(f\psi_n)^+(t) - \int_0^1 H_n(x,t) d(f\psi_n)^-(t) \qquad \ldots(3)$$

and

$$\mathbb{G}_n(f) = f(x) + \int_0^x H_n(x,t) d(f\psi_n)^+(t) - \int_0^1 h_n(x,t) d(f\psi_n)^-(t) \qquad \ldots(4)$$

from (3) and (4), we get:

$$\begin{aligned}\mathbb{G}_n(f) - \mathbb{g}_n(f) &= \int_0^1 \Big[[H_n(x,t) - h_n(x,t)] d(f\psi_n)^+(\mathrm{t}) + \int_0^1 [H_n(x,t) - h_n(\mathrm{x},t)] d(f\psi_n)^-(\mathrm{t}) \\ &= \int_0^1 (1-(x-t)^2)^n d(f\psi_n)^+(t) + \int_0^1 (1-(x-t)^2)^n d(f\psi_n)^-(t)\end{aligned}$$

since $(f\psi_n)^+(x) = \frac{1}{2}\left(V_{f\psi_n}(x) + (f\psi_n)(x)\right)$ and $(f\psi_n)^-(x) = \frac{1}{2}\left(V_{f\psi_n}(x) - f\psi_n(x)\right)$, then:

$$\begin{aligned} d(f\psi_n)^+(x) &= \tfrac{1}{2}\left(dV_{f\psi_n}(x) + d(f\psi_n)(x)\right) \\ d(f\psi_n)^-(x) &= \tfrac{1}{2}\left(dV_{f\psi_n}(x) - d(f\psi_n)(x)\right) \\ &= \int_0^1 (1-(x-t)^2)^n \left\{\tfrac{1}{2} dV_{f\psi_n}(t) + \tfrac{1}{2} df\psi_n(t) + \tfrac{1}{2} dV_{f\psi_n}(t) - \tfrac{1}{2} df\psi_n(t)\right\} \\ &= \int_0^1 (1-(x-t)^2)^n \, dV_{f\psi_n}(t). \quad \blacksquare\end{aligned}$$

We will prove the direct and inverse theorems of best one-sided multiplier approximation of $f \in \mathrm{L}_{p,\psi_n}(X), X = [0,1]$, using the previous operators.

4. Main Results

The main results of this article, which are termed as the direct and inverse theorems, will be stated and proved in this section.

Theorem 4.1 (The Direct Theorem). Let $f \in L_{p,\psi_n}(X), X = [0,1], n \geqslant 2, n \in N$, then:

$$\tilde{E}_n(f)_{p,\psi_n} \leq C_2 \tau_1 \left(V_{f,\frac{1}{\sqrt{n}}}\right)_{p,\psi_n}$$

where C_2 is a constant positive independent of p and n.

Proof. Define $\left[n^{1/2}\right] = \max\left\{q \in z \mid q \leq n^{1/2}\right\}$. Also, we introduce equidistant partition $I = [0,1]$, then $0 = t_0 < t_1 < t_2 < \cdots < t_{[n^{1/2}]+1} = 1$ is a partition of I, with $t_j = \frac{j}{[n^{1/2}+1]}, j = 0,1,\cdots,\left[n^{1/2}\right]+1$

From Lemma (3.6), we have:

$$\mathbb{G}_n(f) - \mathbb{g}_n(f) = \int_0^1 (1-(x-t)^2)^n dV_{f\psi_n}(t)$$

Using the following inequality:

$$(1-(x-t)^2)^n \leq e^{-n(x-t)^2}, \forall\, x,t \in [0,1]$$

Assume $x \in [t_k, t_{k+1}]$, we get:

$$\begin{aligned}\mathbb{G}_n(f) - \mathbb{g}_n(f) &\leq \int_0^1 e^{-n(x-t)^2} dV_{f\psi_n}(t) \\ &\leq \sum_{j=0}^{k-1} \mathrm{e}^{-n(x-t_{j+1})^2} \mathrm{V}_{t_j}^{t_{j+1}}(f\psi_n) + V_k^{t_{k+1}}(f\psi_n) + \sum_{j=\mathrm{k}+1}^{[n^{1/2}]} \mathrm{e}^{-n(x-t_j)} \mathrm{V}_{t_j}^{t_{j-1}}(f\psi_n)\end{aligned}$$

$$\mathbb{G}_n(f) - \mathbb{g}_n(f) \le \sum_{j=0}^{[n^{1/2}]} e^{-n\left(\frac{j}{[n^{1/2}]+1}\right)^2} \omega_1\left(V_{f,}, x, \frac{2(i+2)}{[n^1/2]+1}\right)_{p,\psi_n}$$

$$\left(\int_X |(\mathbb{G}_n(f) - \mathbb{g}_n(f)\psi_n)|^p \, dx\right)^{1/p} \le \sum_{j=0}^{[n^{1/2}]} e^{-n\left(\frac{j}{[n^{1/2}]+1}\right)^2} \left(\int_X \left|\omega_1\left(V_{f,}, x, \frac{2(i+2)}{[n^1/2]+1}\right)_{p,\psi_n}\right|^p dx\right)^1$$

$$\|\mathbb{G}_n(f) - \mathbb{g}_n(f)\|_{p,\psi_n} \le \sum_{j=0}^{[n^{1/2}]} \frac{1}{e^{i^2/4}} \tau_1\left(V_f, \frac{2(i+2)}{[n^{1/2}]+1}\right)_{p,\psi_n}$$

From Lemma (3.3), we get:

$$\|\mathbb{G}_n(f) - \mathbb{g}_n(f)\|_{p,\psi_n} \le \sum_{j=0}^{\infty} \left(\frac{2(i+2)}{e^{i^2/4}}\right) \tau_1\left(V_f, \frac{1}{\sqrt{n}}\right)_{p,\psi_n}$$

$$= C_2 \tau_1\left(V_f, \frac{1}{\sqrt{n}}\right)_{p,\psi_n}. \quad \blacksquare$$

Theorem 4.2 (The Converse Theorem). Let $f \in L_{p,\psi_n}(X), X = [0,1], n \ge 2, n \in N$, then:

$$\tau_1\left(V(f), \frac{1}{\sqrt{n}}\right)_{p,\psi_n} \le C_1 \tilde{E}_n(f)_{p,\psi_n}, 1 \le p < \infty$$

C_1 is a positive constant independent of p and n.

Proof. For all $n \ge 2$, we have $\frac{1}{2e} \le \left(1 - \frac{1}{n}\right)^n$, for $1 \le p < \infty$, then from Lemma (3.3), we have:

$$\tau\left(V(f), \frac{1}{\sqrt{n}}\right)_{p,\psi_n} = \left\| V_{x-\frac{1}{2\sqrt{n}}}^{x+\frac{1}{2\sqrt{n}}}(f, .)\right\|_{p,\psi_n}$$

$$= \left(\int_X \left| V_{x-\frac{1}{2\sqrt{n}}}^{x+\frac{1}{2\sqrt{n}}}(f\psi_n)(x)\right|^p dx\right)^{1/p}$$

$$= \left(\int_X \left\{\int_{x-\frac{1}{2\sqrt{n}}}^{x+\frac{1}{2\sqrt{n}}} dV(f\psi_n)(t)\right\}^p dx\right)^{1/p} \frac{1}{2e}$$

$$\le \left(1 - \frac{1}{n}\right)^n, \forall n \ge 2$$

$$1 \le 2e\left(1 - \frac{1}{n}\right)^n, \forall n \ge 2$$

$$\tau_1\left(Vf, \frac{1}{\sqrt{n}}\right)_{p,\psi_n} \le \left(\int_0^1 \left\{\int_{x-\frac{1}{\sqrt{n}}}^{x+\frac{1}{\sqrt{n}}} 2e - \frac{1}{n} dV_{f\psi_n}(t)\right\} dx\right)^{1/p}$$

Since $x - \frac{1}{\sqrt{n}} \le t \le x + \frac{1}{\sqrt{n}}$, then:

$$\frac{1}{\sqrt{n}} - x \ge -t \ge -\frac{1}{\sqrt{n}} - x$$

$$\frac{1}{\sqrt{n}} \ge x - t \ge \frac{-1}{\sqrt{n}}$$

$$-\frac{1}{\sqrt{n}} \le x - t \le \frac{1}{\sqrt{n}}$$

$$|x - t| \le \frac{1}{\sqrt{n}}$$

$$(x - t)^2 \le \frac{1}{n}$$

$$-(x - t)^2 \ge -\frac{1}{n}$$

$$1 - (x - t)^2 \ge 1 - \frac{1}{n}$$

$$(1 - (x - t)^2)^n \ge \left(1 - \frac{1}{n}\right)^n$$

from Lemma 3.5 (c), we get:

$$\tau_1\left(Vf, \frac{1}{\sqrt{n}}\right)_{p,\psi_n} \le c_1\left(\int_0^1 \left\{\int_0^1 (1 - (x - t)^2)^n dV_{f\psi_n}(t)\right\} dx\right)^{1/p}$$

$$\le c_1 \|\mathbb{G}_n(f)(\cdot) - \mathbb{g}_n(f)(\cdot)\|_{P,\psi_n}$$

$$\le c_1 \tilde{E}_n(f)_{p,\psi_n}. \quad \blacksquare$$

Corollary 4.1. Let $f \in L_{p,\psi_n}(X)$, $X = [0, 1]$, $p \ge 1$, then $\tau_1\ (Vf, \delta)\ \ p, \psi_n = O(\delta)$, $\delta \longrightarrow 0$ is equivalent to $\tilde{E}_n(f)_{p,\psi_n} = O\left(n^{-\frac{1}{2}}\right), n \longrightarrow \infty$.

References

[1] Oleksandr M.; "One-sided weighted approximation". M.Sc. Thesis, Department of Mathematics, University of Manitoba, 2008.
[2] Motornyi V. P.; Pas'ko A.N.; "On the best one-sided approximation of some classes of differentiable functions in L_1-space". 10(1-2): 159-169, 2010.
[3] Rensuoli; Yong ping L.; "The best m-term one-sided approximation of Besor classes by the trigonometric Polynomials". China Advances in Pure Mathematics: 183-184, 2012.
[4] Huseyin K.; Ramazan A.; "Averaged modulus of smoothness and one-sided monotone approximation in or Licz spaces". Article in Hacettepe Journal of Mathematics and Statistics, January 2017.
[5] Sahib K.; Naseif J.; Ali H.; "Best multiplier approximation of unbundled periodic functions in $L_{p,\psi_n}(B)$, $B = [0,2\pi]$ using Discrete linear positive operators". Baghdad Science Journal: 1-5, 2020.
[6] Burkhard L.; "Operators for one-sided approximation by algebraic polynomials". Journal of Approximation Theory, 54: 169-179, 1988.
[7] Vladislav K.; Dzyadyk I.; Shevchuk A.; "Theory of uniform approximation of functions by polynomials". 2008.
[8] Hardy G.H.; "Divergent series". Oxford University Press, New York, 1949.