Al-Nahrain Journal of Science
ANJS, Vol.25 (4), December, 2022, pp. 63-67

Fifth Order Improved Runge-Kutta Nystrom Method Using Trigonometrically-Fitting for Solving Oscillatory Problems

Waleed J. Hasan and Kasim A. Hussain*

Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq

Article's Information

Received:
16.09.2022
Accepted:
08.11.2022
Published:
31.12.2022

Keywords:
Trigonometrically-fitted
Second-order initial value problems
Oscillating solutions
Runge-Kutta Nystrom methods

Abstract

In this paper, the Trigonometrically Fitted Improved Runge-Kutta Nystrom method is proposed as a novel method with four stages and fifth order for solving oscillatory problems. This method is intended to integrate second-order initial value problems using the trigonometrically fitting approach. To increase the method's accuracy, the principal frequency of the problem $w \in \mathbb{R}$, is used. It is discovered that the new method is more precise when compared with the other existing Runge-Kutta Nystrom and IRKN5 methods. To show how well the TFIRKN5 method works, test problems for second-order ordinary differential equations (ODEs) are solved. The numerical outcomes show that the novel approach outperforms methods that have already been published.

DOI: 10.22401/ANJS.25.4.11
Corresponding author: kasimabbas@uomustansiriyah.edu.iq

1. Introduction

The general form of special second-order ODEs is as follows:

$$u''(t) = f(t,u), u(t_0) = u_0, u'(t_0) = u'_0, \qquad \ldots(1)$$

which contains oscillatory character in the solutions. This issue often arises in a number of applied science disciplines, including quantum chemistry, quantum physics, astrophysics, and others [1,2]. The traditional method of solving (1) is transformed into a system of 1st-order ODEs and hence use of an appropriate technique [3]. However, tackling the problem directly using Runge-Kutta Nystrom methods is more efficient [4,5,6]. In 2012, Rabiei et al. [7] built Improved Runge-Kutta Nystrom (IRKN) method to solve second IVPs by introducing the novel terms q_{-i}, which picks the $q_i, i \geq 2$ from its prior steps. Exponentially and trigonometrically fitted techniques are used to develop new methods to solve oscillatory problems. Two explicit two-derivative RKN techniques are constructed in [8], one using trigonometric fitting and the other using exponential fitting. In [9] developed an explicit trigonometrically fitted RKN approach using the Simos methodology. In addition, Demba et al. in 2020 [10] derived an embedded RKN technique with an exponentially-fitted approach to resolve oscillatory second-order IVPs.

In order to resolve oscillatory issues, this study derives Trigonometrically-Fitted fifth order Improved Runge-Kutta Nystrom (TFIRKN5) approach. The TFIRKN5 technique is derived in Section 2. In Section 3, test problems and numerical comparisons with other methods are shown to demonstrate the efficacy of newly proposed approach. In Section 4, conclusions are offered.

2. The Derivation of TFIRKN5 Method

The following is the general form of the s-stages IRKN approach for solving (1), [7]:

$$u_{n+1} = u_n + \frac{3h}{2}u'_n + \frac{h}{2}u'_{n-1} + h^2 \sum_{i=2}^{s} d_i(q_i - q_{-i}), \qquad \ldots(2)$$

$$u'_{n+1} = u'_n + h(b_1 q_1 - b_{-1} q_{-1} + \sum_{i=2}^{s} b_i(q_i - q_{-i}), \qquad \ldots(3)$$

such that:

$$q_1 = f(t_n, u_n), \qquad \ldots(4)$$

$$q_{-1} = f(t_{n-1}, u_{n-1}), \qquad \ldots(5)$$

$$q_i = f\left(t_n + c_i h, u_n + hc_i u'_n + h^2 \sum_{j=1}^{i-1} a_{ij} q_j\right), \qquad \ldots(6)$$

$$q_{-i} = f\left(t_{n-1} + c_i h, u_{n-1} + hc_i u'_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} q_{-j}\right) \qquad \ldots(7)$$

where c_i, b_i, d_i, b_{-1} and a_{ij} are real numbers and $i,j = 1,2,\ldots,s$. The following is the corresponding Butcher tableau for the IRKN method (2)-(7) in Table 1:

Table 1. Butcher tableau of IRKN method.

0					
c_2	a_{21}				
c_3	a_{31}	a_{32}			
.	.	.	.		
.	.	.		.	
.	.	.			.
c_s	a_{s1}	a_{s2}	…	a_{ss-1}	
b_{-1}	b_1	b_2	…	b_{ss-1}	b_s
		d_2	…	d_{ss-1}	d_s

The following provides an illustration of the general structure of the four-stages of the IRKN5 method:

$u_{n+1} = u_n + \frac{3h}{2}u'_n + \frac{h}{2}u'_{n-1} + h^2(d_2(q_2 - q_{-2}) + d_3(q_3 - q_{-3})),$ ...(8)

$u'_{n+1} = u'_n + h(b_1 q_1 - b_{-1}q_{-1} + b_2(q_2 - q_{-2}) + b_3(q_3 - q_{-3})),$ ...(9)

where:

$q_1 = f(t_n, u_n),$ ...(10)

$q_{-1} = f(t_{n-1}, u_{n-1}),$ ...(11)

$q_2 = f(t_n + c_2 h, u_n + hc_2 u'_n + h^2 a_{21} q_1),$ ...(12)

$q_{-2} = f(t_n + c_2 h, u_{n-1} + hc_2 u'_{n-1} + h^2 a_{21} q_{-1})$ ...(13)

$q_3 = f(t_n + c_3 h, u_n + hc_3 u'_n + h^2(a_{31}q_1 + a_{32}q_2),$ ...(14)

$q_{-3} = f(t_n + c_3 h, u_{n-1} + hc_3 u'_{n-1} + h^2(a_{31}q_{-1} + a_{32}q_{-2}),$ ...(15)

$q_4 = f(t_n + c_4 h, u_n + hc_4 u'_n + h^2(a_{41}q_1 + a_{42}q_2 + a_{43}q_3),$ ...(16)

$q_{-4} = f(t_n + c_4 h, u_{n-1} + hc_4 u'_{n-1} + h^2(a_{41}q_{-1} + a_{42}q_{-2} + a_{43}q_{-3}),$ ...(17)

If the IRKN5 method (8)-(17) is used to precisely integrate $u(t_n)$, then we have:

$u_n = u(t_n) = e^{i\omega t_n},$ ...(18)

$u'_n = i\omega e^{i\omega t_n},$ ...(19)

$u''_n = -\omega^2 e^{i\omega t_n} = f(t_n, u_n),$ ...(20)

$u_{n-1} = u(t_{n-1}) = e^{i\omega(t_n - h)},$ ...(21)

$u'_{n-1} = i\omega e^{i\omega(t_n - h)},$ ...(22)

$u''_{n-1} = -\omega^2 e^{i\omega(t_n - h)} = f(t_{n-1}, u_{n-1}),$ ...(23)

$u_{n+1} = e^{i\omega(t_n + h)}.$ ...(24)

Equations (18)-(24) can be substituted into equation (8)-(17) and using the Euler formula $e^{iv} = \cos(v) + i\sin(v)$. We acquire the trigonometrically-fitting order conditions by separating the real and imaginary parts:

$\cos(v) = 1 + \frac{1}{2}\, v\sin(v) - v^2 \sum_{i=2}^{3} d_i(\cos(c_i v) - \cos(c_i v - v)),$ ...(25)

$\sin(v) = \frac{3}{2}v - \frac{1}{2}\, v\cos(v) - v^2 \sum_{i=2}^{3} d_i(\sin(c_i v) - \sin(c_i v - v)),$ ...(26)

$\cos(v) = 1 - b_{-1}\, v\sin(v) - v\sum_{i=2}^{3} b_i(\sin(c_i v) - \sin(c_i v - v)),$ ...(27)

$\sin(v) = b_1 v - b_{-1}\, v\cos(v) - v\sum_{i=2}^{3} b_i(\cos(c_i v) - \cos(c_i v - v)).$ ...(28)

where $v = \omega h$. The IRKN5 approach, which consists of four-stages of order 5, is described in this paper as follows in Table 2, [7]:

Table 2. Four-stage fifth-order IRKN5 method.

0				
$\frac{1}{4}$	$\frac{1}{32}$			
$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{7}{24}$		
$\frac{3}{4}$	$\frac{4}{32}$	0	$\frac{5}{32}$	
$\frac{1}{45}$	$\frac{46}{45}$	$-\frac{1}{15}$	$-\frac{1}{10}$	$\frac{29}{45}$
		$\frac{49}{180}$	$\frac{7}{180}$	$\frac{19}{180}$

To construct trigonometrically-fitted IRKN5 method of four-stage fifth-order. The following additional order conditions for the IRKN5 method are used;

$b_2 + b_3 + b_4 + b_{-1} = \frac{1}{2},$ ...(29)

$b_2 c_2 + b_3 c_3 + b_4 c_4 = \frac{5}{12},$ ...(30)

$b_2 c_2^3 + b_3 c_3^3 + b_4 c_4^3 = \frac{31}{120},$ ...(31)

$d_2 c_2^2 + d_3 c_3^2 + d_4 c_4^2 = \frac{31}{360},$ ...(32)

Solving equations (25) - (28) and (29) - (32) using the parameters of IRKN5 method in Table 2 for $b_{-1}, b_1, b_2, b_3, b_4, d_2, d_3$ and d_4, we obtain the solution as follows using Maple package:

$$b_{-1} = -\frac{\left(29\,v\sin\left(\frac{1}{4}v\right) + 29\,v\sin\left(\frac{3}{4}v\right) - 13v\sin\left(\frac{1}{2}v\right) + 45\cos(v) - 45\right)}{45v\left(-3\sin\left(\frac{1}{4}v\right) - 3\sin\left(\frac{3}{4}v\right) + 4\sin\left(\frac{1}{2}v\right) + \sin(v)\right)}, \quad ...(33)$$

$$b_1 = \frac{1}{45v\left(-3\sin\left(\frac{1}{4}v\right) - 3\sin\left(\frac{3}{4}v\right) + 4\sin\left(\frac{1}{2}v\right) + \sin(v)\right)} \Big(-90\cos\left(\frac{1}{4}v\right)\cos(v) + 90\cos\left(\frac{1}{4}v\right) + 90\cos\left(\frac{3}{4}v\right)\cos(v) - 135\sin\left(\frac{1}{4}v\right)\sin(v) - 135\sin\left(\frac{3}{4}v\right)\sin(v) + 13v\cos(v)\sin\left(\frac{1}{2}v\right) + 180\sin\left(\frac{1}{2}v\right)\sin(v) + 45\sin^2(v) - 29v\cos(v)\sin\left(\frac{3}{4}v\right) - 30v\cos\left(\frac{3}{4}v\right)\sin(v) + 146v\cos\left(\frac{1}{4}v\right)\sin\left(\frac{1}{2}v\right) - 45\cos^2(v) + 30v\cos\left(\frac{1}{4}v\right)\sin(v) + 45\cos(v) - 148v\cos\left(\frac{1}{4}v\right)\sin\left(\frac{1}{4}v\right) - 148v\cos\left(\frac{1}{4}v\right)\sin\left(\frac{3}{4}v\right) - 146v\cos\left(\frac{3}{4}v\right)\sin\left(\frac{1}{4}v\right) + 148v\cos\left(\frac{3}{4}v\right)\sin\left(\frac{1}{4}v\right) + 148v\cos\left(\frac{3}{4}v\right)\sin\left(\frac{3}{4}v\right) - 29v\cos(v)\sin\left(\frac{1}{4}v\right) - 90\cos\left(\frac{3}{4}v\right)\Big), \quad ...(34)$$

$$b_2 = -\frac{1}{90v\left(-3\sin\left(\frac{1}{4}v\right) - 3\sin\left(\frac{3}{4}v\right) + 4\sin\left(\frac{1}{2}v\right) + \sin(v)\right)} \Big(-148v\sin\left(\frac{1}{4}v\right) + 225 - 148v\sin\left(\frac{3}{4}v\right) + v\sin(v) - 255\cos(v) + 69v\sin\left(\frac{1}{2}v\right)\Big), \quad ...(35)$$

$$b_3 = -\frac{1}{90v\left(-3\sin\left(\frac{1}{4}v\right) - 3\sin\left(\frac{3}{4}v\right) + 4\sin\left(\frac{1}{2}v\right) + \sin(v)\right)} \Big(77v\sin\left(\frac{1}{4}v\right) - 180 + 13\,v\sin(v) + 180\cos(v) + 77v\sin\left(\frac{3}{4}v\right)\Big), \quad ...(36)$$

$$b_4 = \frac{1}{90v\left(-3\sin\left(\frac{1}{4}v\right) - 3\sin\left(\frac{3}{4}v\right) + 4\sin\left(\frac{1}{2}v\right) + \sin(v)\right)} \Big(223v\sin\left(\frac{1}{2}v\right) + 59v\sin(v) + 45\cos(v) - 148v\sin\left(\frac{3}{4}v\right) - 45 - 148v\sin\left(\frac{1}{4}v\right)\Big), \quad ...(37)$$

$$d_2=-\frac{1}{180\left(v^2\left(2\sin\left(\frac{1}{4}v\right)+2\sin\left(\frac{3}{4}v\right)-5\sin\left(\frac{1}{2}v\right)\right)\left(\cos\left(\frac{1}{2}v\right)-\cos\left(\frac{3}{4}v\right)\right)\right)}\Big(-180\sin\left(\tfrac{1}{4}v\right)+810\sin\left(\tfrac{1}{2}v\right)-180\sin\left(\tfrac{3}{4}v\right)+90v\sin\left(\tfrac{1}{4}v\right)\sin(v)+90v\sin\left(\tfrac{3}{4}v\right)\sin(v)-405v\sin\left(\tfrac{1}{2}v\right)\sin(v)+270v\cos\left(\tfrac{3}{4}v\right)+124v^2\sin\left(\tfrac{1}{2}v\right)\cos\left(\tfrac{1}{4}v\right)-124v^2\sin\left(\tfrac{1}{2}v\right)\cos\left(\tfrac{3}{4}v\right)+90v\cos(v)\cos\left(\tfrac{1}{4}v\right)-90v\cos(v)\cos\left(\tfrac{3}{4}v\right)+180\sin\left(\tfrac{1}{4}v\right)\cos(v)+180\sin\left(\tfrac{3}{4}v\right)\cos(v)-180\sin(v)\cos\left(\tfrac{3}{4}v\right)-810\sin\left(\tfrac{1}{2}v\right)\cos(v)+180\sin(v)\cos\left(\tfrac{1}{4}v\right)-270v\cos\left(\tfrac{1}{4}v\right)\Big), \quad \ldots(38)$$

$$d_3=-\frac{1}{180\left(v^2\left(2\sin\left(\frac{1}{4}v\right)+2\sin\left(\frac{3}{4}v\right)-5\sin\left(\frac{1}{2}v\right)\right)\left(\cos\left(\frac{1}{2}v\right)-\cos\left(\frac{3}{4}v\right)\right)\right)}\Big(360\sin\left(\tfrac{1}{4}v\right)+360\sin\left(\tfrac{3}{4}v\right)-180v\sin\left(\tfrac{1}{4}v\right)\sin(v)-180v\sin\left(\tfrac{3}{4}v\right)\sin(v)+124v^2\sin\left(\tfrac{1}{4}v\right)\cos\left(\tfrac{1}{4}v\right)-124v^2\sin\left(\tfrac{1}{4}v\right)\cos\left(\tfrac{3}{4}v\right)+124v^2\sin\left(\tfrac{3}{2}v\right)\cos\left(\tfrac{1}{4}v\right)-124v^2\sin\left(\tfrac{3}{2}v\right)\cos\left(\tfrac{3}{4}v\right)+225v\cos(v)\cos\left(\tfrac{1}{4}v\right)-225v\cos(v)\cos\left(\tfrac{3}{4}v\right)-360\sin\left(\tfrac{1}{4}v\right)\cos(v)-360\sin\left(\tfrac{3}{4}v\right)\cos(v)+675v\cos\left(\tfrac{3}{4}v\right)+450\sin(v)\cos\left(\tfrac{1}{4}v\right)-450\sin(v)\cos\left(\tfrac{3}{4}v\right)-675v\cos\left(\tfrac{1}{4}v\right)\Big), \quad \ldots(39)$$

$$d_4=-\frac{1}{180\left(v^2\left(2\sin\left(\frac{1}{4}v\right)+2\sin\left(\frac{3}{4}v\right)-5\sin\left(\frac{1}{2}v\right)\right)\left(\cos\left(\frac{1}{2}v\right)-\cos\left(\frac{3}{4}v\right)\right)\right)}\Big(180\sin\left(\tfrac{1}{4}v\right)+180\sin\left(\tfrac{3}{4}v\right)-90v\sin\left(\tfrac{3}{4}v\right)\sin(v)-90v\sin\left(\tfrac{1}{4}v\right)\sin(v)+124v^2\sin\left(\tfrac{1}{2}v\right)\cos\left(\tfrac{1}{4}v\right)-124v^2\sin\left(\tfrac{1}{2}v\right)\cos\left(\tfrac{3}{4}v\right)+90v\cos(v)\cos\left(\tfrac{1}{4}v\right)-90v\cos(v)\cos\left(\tfrac{3}{4}v\right)-180\sin\left(\tfrac{1}{4}v\right)\cos(v)-18\sin\left(\tfrac{3}{4}v\right)\cos(v)-90\sin\left(\tfrac{1}{2}v\right)-180\sin(v)\cos\left(\tfrac{3}{4}v\right)+45v\sin\left(\tfrac{1}{2}v\right)\sin(v)+270v\cos\left(\tfrac{3}{4}v\right)+90\sin\left(\tfrac{1}{2}v\right)\cos(v)+180\sin(v)\cos\left(\tfrac{1}{4}v\right)-270v\cos\left(\tfrac{1}{4}v\right)\Big), \quad \ldots(40)$$

Thus, for the small value of v, the corresponding Taylor series expansions are presented as follows:

$$b_{-1}=\frac{1}{45}-\frac{1}{288}v^2-\frac{401}{1935360}v^4-\frac{10987}{928972800}v^6-\frac{2645119}{3923981107200}v^8-\frac{6584928269}{171399494762496000}v^{10}+\cdots,$$

$$b_1=\frac{46}{45}-\frac{1}{288}v^2-\frac{401}{1935360}v^4-\frac{15901}{132710400}v^6+\frac{2896769}{3923981107200}v^8-\frac{21725682989}{171399494762496000}v^{10}+\cdots,$$

$$b_2=-\frac{1}{15}+\frac{5}{576}v^2+\frac{401}{774144}v^4+\frac{10987}{371589120}v^6+\frac{2645119}{1569592442880}v^8+\frac{6584928269}{68559797904998400}v^{10}+\cdots,$$

$$b_3=-\frac{1}{10}-\frac{1}{144}v^2-\frac{401}{967680}v^4-\frac{10987}{464486400}v^6-\frac{2645119}{1961990553600}v^8-\frac{6584928269}{85699747381248000}v^{10}+\cdots,$$

$$b_4=\frac{29}{45}+\frac{1}{576}v^2+\frac{401}{3870720}v^4+\frac{10987}{1857945600}v^6+\frac{2645119}{7847962214400}v^8+\frac{6584928269}{342798989524992000}v^{10}+\cdots,$$

$$d_2=\frac{49}{180}-\frac{7}{1152}v^2+\frac{419}{1105920}v^4-\frac{186973}{3715891200}v^6+\frac{101902643}{15695924428800}v^8-\frac{1841000153}{2176501520793600}v^{10}+\cdots,$$

$$d_3=\frac{7}{180}+\frac{7}{720}v^2-\frac{361}{387072}v^4+\frac{116821}{928972800}v^6-\frac{9098489}{560568729600}v^8+\frac{3097835927}{1464952946688000}v^{10}+\cdots,$$

$$d_4=\frac{19}{180}-\frac{7}{1920}v^2+\frac{961}{2580480}v^4-\frac{26701}{530841600}v^6+\frac{4076125}{627836977152}v^8-\frac{579914867461}{685597979049984000}v^{10}+\cdots,$$

3. Numerical Results and Discussion

Four different problems were used to test in this section. Numerical outcomes of the suggested method are contrasted with the well-known RKN algorithms of the same-order. The maximum absolute error is.

$$MaxError=max(|u(t_n)-u_n|),$$

where $u(t_n)$ is the exact solution and u_n is the numerical solution. Figures 1 - 4 reveal the competence curves of $Log_{10}(MaxError)$ against the step length h. The following abbreviations were utilized to carry out the numerical experiments:

- **TFIRKN5:** the trigonometrically-fitted four-stage fifth-order IRKN5 method derived in this chapter.
- **IRKN5:** the four-stage RKN method of order five given in [7].
- **RKN5V:** RKN technique of order 5 presented in [11].
- **RKN5S:** RKN approach of order 5 offered in [12].
- **RKN5M:** RKN technique of order 5 given in [13].

Problem 1, [14].

$$u''(t)=-u(t)+t, u(0)=1, u'(t)=2.$$

The exact solution: $u(t)=\sin(t)+\cos(t)+t.$

Problem 2, [15].

$$u_1''(t)=-\frac{101}{2}u_1(t)+\frac{99}{2}u_2(t)+\frac{93}{2}cos(2t)-\frac{99}{2}sin(2t),\ u_1(0)=0, u_1'(0)=-10.$$

$$u_2''(t)=\frac{99}{2}u_1(t)-\frac{101}{2}u_2(t)+\frac{93}{2}sin(2t)-\frac{99}{2}cos(2t),\ u_2(0)=1, u_2'(0)=12.$$

The exact solution:

$$u_1(t)=-\cos(10t)-\sin(10t)+\cos(2t),$$
$$u_2(t)=\cos(10t)+\sin(10t)+\sin(2t).$$

Problem 3, [13].

$$u_1''(t)=-400\,u_1(t)+(400+0.0025)e^{-0.05t},$$
$$u_1(0)=1.1,\ u_1'(0)=-0.05,$$
$$u_2''(t)=-400\,u_2(t)+(400+0.0025)e^{-0.05t},$$
$$u_2(0)=1.0,\ u_2'(0)=1.95$$

The exact solution:

$u_1(t) = 0.1\cos(20\,t) + e^{-0.05\,t}$,
$u_1(t) = 0.1\sin(20\,t) + e^{-0.05\,t}$.

Problem 4, [16].

$u_1''(t) + u_1(t) = 0.001\ \cos(t)$,
$u_1(0) = 1, u_1'(0) = 0$,
$u_2''(t) + u_2(t) = 0.001\ \sin(t)$,
$u_2(0) = 0, u_2'(0) = 0.9995$.

The exact solution:

$u_1(t) = \cos(t) + 0.0005\,t\sin(t)$,
$u_2(t) = \sin(t) - 0.0005\,t\cos(t)$.

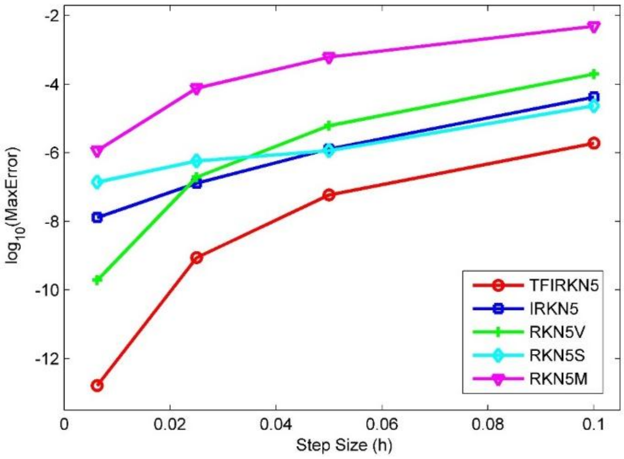

Figure 1. Competence curves with step size $h = \frac{0.1}{2^r}, r = 0,1,2,4$ for problem1.

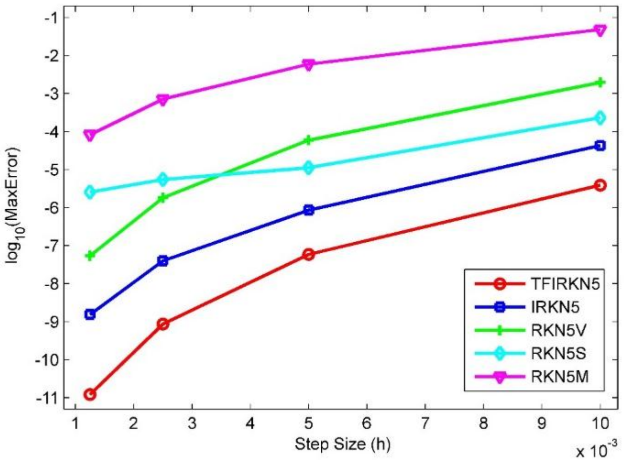

Figure 2. Competence curves with step size $h = \frac{0.01}{2^r}, r = 0,1,2,3$ for problem 2.

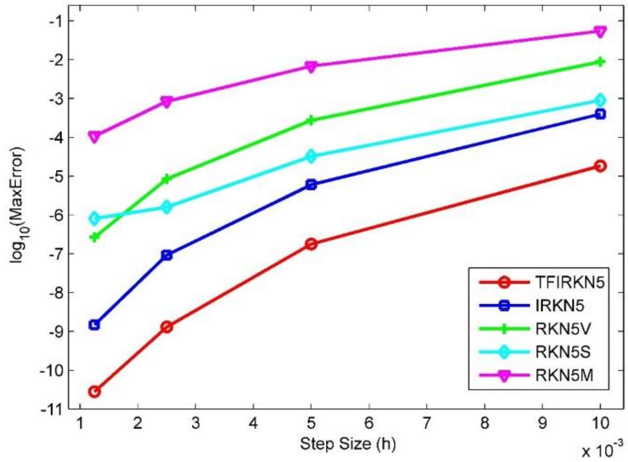

Figure 3. Competence curves with step size $h = \frac{0.01}{2^r}, r = 0,1,2,3$ for problem 3.

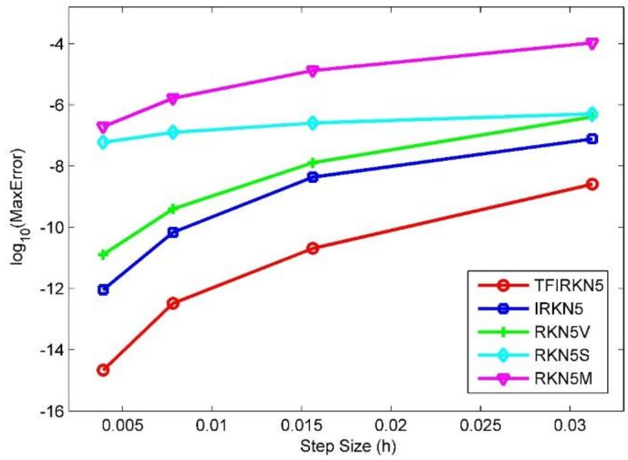

Figure 4. Competence curves with step size $h = \frac{0.03125}{2^r}, r = 0,1,2,3$ for problem 4.

The numerical results for solving second-order ODEs with the integration interval [0,1000] are shown in Figures 1 - 4. Which represent the maximum error for each method. The TFIRKN5 approach is contrasted with the current IRKN5, RKN5V, RKN5S, and RKN5M approaches. The maximum error obviously lowers as the number of step sizes h grows smaller. The suggested approach, TFIRKN5, has the least maximum error of the four ways while solving four numerical tests. In Figure 1, we demonstrate the efficiency of the proposed approach to the inhomogeneous problem, which shows the approaches of TFIRKN5, IRKN5, RKN5S, RKN5V, and RKN5M, in decreasing order of effectiveness. In Figures 2 - 4, we show how well the five approaches work for the nonlinear nonhomogeneous system. We see that in terms of accuracy, the new TFIRKN5 strategy outperforms IRKN5, RKN5S, RKN5V, and RKN5M ways.

4. Conclusions

In this article, we derived a trigonometrically-fitted TFIRKN5 technique for solving second-order ODEs with periodic solutions based on IRKN5 approach of order 5 with 4 stages in [9]. The numerical results for solving problems 1 to 4, as presented in Figures 1 - 4, reveal that the novel TFIRKN5 approach has a reduced maximum error norm than the other methods that are currently used in the literature. We come to the conclusion that for solving second-order ODEs with periodic solutions, the TFIRKN5 approach is more precise and effective.

Acknowledgement
The authors thank Mustansiriyah University (www.uomustansiriyah.edu.iq), College of Science, Department of Mathematics for supporting this work.

Conflicts of Interest
The authors declare that there is no conflict of interest.

References
[1] Hairer E; Nrsett S. P. and Wanner G.; "Solving ordinary differential equations I: Nonstiff problems", Springer; Berlin, 1993.
[2] Butcher J. C.; "Numerical methods for ordinary differential equations", 2nd ed., Wiley; New York, 2008.
[3] Salih M.; Ismail F. and Senu N.; "Phase fitted and amplification fitted of Runge-Kutta-Fehlberg method of order 4 (5) for solving oscillatory problems", Baghdad Science Journal, 17(2): 689-689, 2020.
[4] Franco J. M.; "Embedded pairs of explicit ARKN methods for the numerical integration of perturbed oscillators", Journal of Computational Methods in Sciences and Engineering, 3(3): 415-424, 2003.
[5] Zhai W.; Fu S.; Zhou T. and Xiu C.; "Exponentially-fitted and trigonometrically-fitted implicit RKN methods for solving $y''(x) = f(x, y)$ ", Journal of Applied Mathematics and Computing, 68(2): 1449-1466, 2022.
[6] Kovalnogov V. N.; Fedorov R. V.; Generalov D. A.; Tsvetova E. V.; Simos T. E. and Tsitouras C.; "On a new family of Runge-Kutta-Nystrom pairs of orders 6(4)", Mathematics, 10(6): 1-15, 2022.
[7] Rabiei F.; Ismail F.; Senu N. and Abasi N.; "Construction of improved Runge-Kutta Nystrom method for solving second-order ordinary differential equations", World Applied Sciences Journal, 20(12): 1685-1695, 2012.
[8] Mohamed T. S.; Senu N.; Ibrahim Z. B. and Nik Long N. M. A.; "Exponentially fitted and trigonometrically fitted two-derivative Runge-Kutta-Nyström methods for solving $y''(x) = f(x, y, y')$ ", Mathematical Problems in Engineering, 2018: 1-19, 2018.
[9] Demba M. A.; Senu N. and Ismail F.; "Trigonometrically-fitted explicit four-stage fourth-order Runge–Kutta–Nyström method for the solution of initial value problems with oscillatory behavior", Global Journal of Pure and Applied Mathematics, 12(1): 67-80, 2016.
[10] Demba M. A.; Kumam P.; Watthayu W. and Phairatchatniyom P.; "Embedded exponentially-fitted explicit Runge-Kutta-Nystrom methods for solving periodic problems", Computations, 8(2): 1-12, 2020.
[11] Van der Houwen P. J. and Sommeijer, B. P.; " Explicit Runge-Kutta-Nyström methods with reduced phase error for computing oscillating solutions", SIAM J. Numer. Anal., 24(3): 595-617, 1987.
[12] Simos T. E.; Dimas E. and Sideridis, A. B.; "A Runge-Kutta-Nyström method for the numerical integration of special second-order periodic initial-value problems", J. Comput. Appl. Math., 51: 317-326, 1994.
[13] Salih M.; Ismail F. and Senu N.; "Fifth order Runge-Kutta Nystrom methods for solving linear second order oscillatory problems", Far East Journal of Applied Mathematics, 95(2): 141-156, 2016.
[14] Senu N.; Lee K. C.; Wan Ismail W. F.; Ahmadian A.; Ibrahim S. N. and Laham M.; "Improved Runge-Kutta method with trigonometrically-fitting technique for solving oscillatory problem", Malaysian Journal of Mathematical Sciences, 15(2): 253-266, 2021.
[15] Hussain K.A. and Abdulnaby Z.E.; "A new two derivative FSAL Runge-Kutta method of order five in four stages", Baghdad Science Journal, 17(1): 161-171, 2020.
[16] Hussain K.A.; "Trigonometrically fitted fifth-order explicit two-derivative Runge-Kutta method with FSAL property", Journal of Physics: Conference Series, 1294: 1-11, 2019