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# About the Existence and Uniqueness Theorem of Fuzzy Random Ordinary Differential Equations

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**Abstract**

Ordinary differential equations that includes stochastic processes in their vector fields are called random ordinary differential equations, that are considered through this work as a Weiner process or also called Brownian motion. In this paper, fuzzy random ordinary differential equations are considered, in which the fuzziness appears in the initial conditions in terms of triangular fuzzy numbers. Such equations are crucial in the theory of random dynamical systems and/or modern control theory and therefore the existence of a unique solution of such equations is of great importance. The statement and the proof of the existence and uniqueness theorem of fuzzy random ordinary differential equations is the main objective of this paper, which is proved using Banach contraction mapping theorem.

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## 1. Introduction

It is commonly known that Zadeh introduced in 1965 the elementary of fuzzy logic through studying several real-life examples and then give alternatively its basic set their operations, [1]. For that, Congxin Wu and Shiji Song in 1996 studied the Cauchy problem of fuzzy differential equations with fuzzy valued mappings of a real variable by using the concept of Hukuhara differentiability (H-differentiability), [6]. Also, Jon Yeoul Park and Hyo Kun Han in 1999 use the method of successive approximations to state and prove the existence and uniqueness theorem of a solution of fuzzy differential equations, [3]. For the fuzzy first-order initial value problem, Buckley and Feuring in 2000 proposed a new solution approach based on the theory of fuzzy logic and fuzzy ordinary differential equations, [2]. Again in 2004, Shiji Song and Congxin Whup studied the fuzzy Cauchy problem and also proposed the concept of level wise continuity to study such equations, [4]. In 2013, Cuilian You solved the general linear fuzzy ordinary differential equations and reducible fuzzy differential equations through performing a substitution of change of variables, [5]. Also, Marin H. Suhniem in 2017 provided a numerical method for solving fuzzy initial value problem based on an artificial neural network for first-order problem considered under generalized H-differentiability, subject to fuzzy initial conditions, [29]. Fadhel et al. studied an proved the in 2021 contraction mapping theorem on partial fuzzy metric spaces as a generalization of the crisp partial metric spaces, [30]. Eidi et al. in 2022 stated and proved two fixed point theorems in fuzzy metric spaces based on different fuzzy contraction mappings, [31].

Fuzzy random ordinary differential equations represent reliable model of the dynamical system's related to real-life phenomena's, in which two kinds of uncertainty or fuzziness appeared and presented in the model; the first one because of the randomness and the second due to the vague notions. They appear to have a shadow existence in the form of stochastic ordinary differential equations with the stochastic process of Itô type, which are used in an extremely wide variety of well-known applications in real life. Fuzzy random ordinary differential equations, in particular, are useful in the study of random dynamical systems and/or non classical control theory, [13].

In 2000, Yuhu Feng [9], discusses the general theory of fuzzy stochastic systems of differential equations, as well as, the existence and uniqueness of solutions. It is proved in 2007 by Fei the existence and uniqueness of solutions for fuzzy random ordinary differential equations with non-Lipschitz coefficients and then the dependence of the fuzzy ordinary differential equations on initial conditions is discussed, [10]. Malinowski in 2009 [11] proved the solution's existence and uniqueness of fuzzy random ordinary differential equation under Lipschitzean right hand side conditions. Again, Malinowski gave two different forms of the solution of fuzzy random ordinary differential equation and applied the concept of successive approximations under a Lipschitzian generalized condition to study the existence and uniqueness theorem to the both types of solutions of such equations. This work was released in 2012. [12].

Unlike stochastic ordinary differential equations, fuzzy random ordinary differential equations may be studied path wise using deterministic calculus, which needs more advanced analytical approaches than those followed in the classical ordinary differential equation theory, [14]. Because of the deriving process appeared in the random ordinary differential equations, which has at most of Hölder continuous sample paths, then the solution is continuously differentiable as a sample path, but the sample path derivative is no more than Hölder continuous in time, after deriving stochastic process, regardless of how smooth the vector field in its original variables. Because of this, solutions of fuzzy random ordinary differential equations are not smooth enough to have Taylor expansions in the usual sense, [15].

Due to the notion of fixed points, there is more than one direction for establishing the existence and uniqueness of a solution to ordinary differential equations. Banach in 1922 established the fundamental principle of fixed point theory, which is known later as Banach contraction mapping principle, [16]. Banach contraction has been refined, expanded, and generalized by numerous authors since it is widely used and valuable technique for solving several mathematical problems (see [17-19]). In this paper, we state and prove the existence and uniqueness theorem of fuzzy random ordinary differential equations using Banach fixed point theorem and based on certain type of stochastic sequence convergence known as converges with probability one (which is also given and defined for equations and inequalities).

The general form of the fuzzy random ordinary differential equation that will be considered in this paper has the form:

$$\tilde{x}'(t,\omega) \overset{[a,b].p.1}{=} f(t,\tilde{x}(t,\omega))$$

with fuzzy initial conditions:

$$\tilde{x}(t_0,\omega) \overset{p.1}{=} \tilde{x}_0$$

where $t_0 \in \mathbb{R}, x_0 \in E^d$, $E^d$ is the set of all nonempty closed and bounded fuzzy subsets of $\mathbb{R}^d$, $d \in \mathbb{N}$, $f: [a,b] \times E^d \longrightarrow E^d$, $\omega$ is the Wiener process and $\tilde{x}(t,\omega)$ is a random fuzzy process. The symbol $\overset{[a,b].p.1}{=}$ means that the probability of the set $\{\omega \in \Omega \mid \tilde{x}(t,\omega) = \tilde{y}(t,\omega), \forall\, t \in [a,b]\}$ equals 1, while $\overset{p.1}{=}$ stands for the probability of the set $\{\omega \in \Omega | \tilde{x}(t_0,\omega) = \tilde{x}_0\}$ also equals 1 at a point $t_0$.

## 2. Preliminaries

To state and prove the existence and uniqueness theorem of fuzzy random ordinary differential equations, the following definitions and theorems are needed.

**Definition 2.1, [20].** If $X$ is a nonempty set and $f: X \longrightarrow X$ be a mapping, then $f$ is said to have a fixed point $x^* \in X$, if $f(x^*) = x^*$.

**Definition 2.2, [20].** A mapping $f: X \longrightarrow X$ defined on a metric space $(X,d)$ is said to be Lipschitzian if there exist a constant $k > 0$ (called Lipschitzn constant), and:

$d(f_x, f_y) \leq kd(x,y)$, for all $x,y \in X$

and when $k < 1$, then $f$ is a contraction.

**Theorem 2.1 (Banach fixed pint theorem), [20].** Let $(X,d)$ be complete metric space and suppose $f: X \longrightarrow X$ be a contraction mapping with contractivity factor $k \in (0,1)$, then $f$ has a unique fixed point $x^* \in X$. Furthermore, $\lim_{n\to\infty} f^n(x) = x^*$, for all $x \in X$ with $d(f^n(x), x^*) \leqslant \frac{k^n}{1-k} d(f(x), x)$.

**Definition 2.3, [21].** A normed space $(X, \|.\|)$ is said to be complete if every Cauchy sequence in $X$ converge to a point in $X$. Complete normed spaces are also called a Banach spaces.

Every Banach space $(X, \|.\|)$ is also a complete metric space $(X, d)$, with metric defined by $d(x,y) = \| x - y \|$. In fuzzy set theory, we need the Housdorff distance between two nonempty bounded subsets $A$ an $B$ of $\mathbb{R}^d$, $d \in \mathbb{N}$, which is defined as follows [11]:

$$d_H(A,B) = \max\left\{\sup_{x\in A} \inf_{y\in B} \|x-y\|, \sup_{x\in B} \inf_{y\in A} \|x-y\|\right\}$$

where $\|.\|$ denotes the usual norm in $\mathbb{R}^n$.

## 3. Related Concepts in Fuzzy Sets and Stochastic Calculus

This work needs some additional advanced concepts in fuzzy set theory and stochastic calculus. These concepts are pointed in this section by starting with the following concept; let compconv $(\mathbb{R}^d)$ be used to represent the family of all compact and convex nonempty fuzzy subsets of $\mathbb{R}^d$, and define the addition and scalar multiplication in compconv $(\mathbb{R}^d)$ using the extension principle followed in fuzzy logic, [11]. It is well known that from literatures for the set of fuzzy triangular numbers, the compconv $(\mathbb{R}^d, d_H)$ becomes a complete and separable metric space, which may be denoted by $E^d = \{\tilde{u} | \mu_{\tilde{u}}: \mathbb{R}^d \longrightarrow [0,1]\}$, where the membership function $\mu_{\tilde{u}}$ of the fuzzy set $\tilde{u}$ satisfies (i) - (iv) below:

(i) $\tilde{u}$ is normal, i.e.; there exists $x_0 \in \mathbb{R}^d$, such that $\mu_{\tilde{u}}(x_0) = 1$.

(ii) $\tilde{u}$ is convex fuzzy set, i.e.; $\mu_{\tilde{u}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{u}}(x), \mu_{\tilde{u}}(y)\}$, for all $x, y \in \mathbb{R}^d$ and $\lambda \in [0,1]$.

(iii) $\tilde{u}$ is upper semicontinuous.

(iv) $[u]^0 = cl\{x \in \mathbb{R}^d \mid \mu_{\tilde{u}}(x) > 0\}$ is compact.

For $\alpha \in [0,1]$, denote $[u]^\alpha = \{x \in \mathbb{R}^d \mid \mu_{\tilde{u}}(x) \geq \alpha\}$. This set will be called an $\alpha$-cut (or $\alpha$-level set) of $\tilde{u}$. For $\tilde{u} \in E^d$, then $[u]^\alpha \in \text{comp conv } (\mathbb{R}^d)$, for every $\alpha \in [0,1]$. Also, it is observed that compact subsets of $\mathbb{R}^d$, which are nonempty can be included in $E^d$ by means of their membership functions.

According to Zadeh's extension principle, and if $g: \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a function, then $g$ can be extended to be defined from $E^d \times E^d$ onto $E^d$ by:

$$g(\tilde{u}, \tilde{v})(z) = \sup_{z=g(x,y)} \{\mu_{\tilde{u}}(x), \mu_{\tilde{v}}(y)\}$$

Also, if $g$ is continuous, then it is well known that $[g(u,v)]^\alpha = g([u]^\alpha, [v]^\alpha)$, for all $\tilde{u}, \tilde{v} \in E^d$, $\alpha \in [0,1]$. In particular, for fuzzy number's addition and scalar multiplication in the space $E^d$, are given by:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \ [cu]^\alpha = c[u]^\alpha$$

where $\tilde{u}, \tilde{v} \in E^d$, $c \in \mathbb{R}$ and $\alpha \in [0,1]$.

Define $D: E^d \times E^d \longrightarrow [0, \infty)$ by:

$$D(\tilde{u}, \tilde{v}) = \sup_{\alpha \in [0,1]} d_H([u]^\alpha, [v]^\alpha)$$

where $d_H$ is the Housdorff metric defined on compconv $(\mathbb{R}^d)$. Also, one may prove that $D$ is a metric on $E^d$. In fact, $(E^d, D)$ is a complete metric space, and for every $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z} \in E^d$ and $c \in \mathbb{R}$, then:

$$D(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}) = D(\tilde{u}, \tilde{v})$$
$$D(\tilde{u} + \tilde{v}, \tilde{w} + \tilde{z}) \leq D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{z})$$
$$D(c\tilde{u}, c\tilde{v}) = |c| D(\tilde{u}, \tilde{v})$$

As a second part of this section related to stochastic calculus, let $(\Omega, F, p)$ be a complete probability space, where $\Omega$ is a sample space, which is the set of all possible outcomes, an event space $F$ which is a set of events, in which the event is a collection of all outcomes in the sample space, and a probability measurable function $p$, which assigns a value between 0 and 1 to each event in the event space $F$.

The random variable is a real valued function $x(\omega)$, $\omega \in \Omega$, which is measurable with respect to the probability function $p$, [23,24]. A stochastic process is a family of random variables $x(t,\omega)$ of two variables, which are assumed to be $t \in [t_0, T] \subset [0, \infty)$ and $\omega \in \Omega$ on a common probability space, [25,26]. Wiener process or Brownian motion, denoted by $W_t$, for all $t \in [0, \infty)$, is a stochastic process, that satisfies:

1. $p(\{\omega \in \Omega \mid W_0(\omega) = 0\} = 1$.
2. For $0 < t_0 < t_1 < \cdots < t_n$ the increments $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots; W_{t_n} - W_{t_{n-1}}$ are independent.
3. For an arbitrary $t$ and $h > 0$, $W_{t+h} - W_t$ has the normal distribution with mean 0 and variance $h$.

The convergence of the sequence of random variable $\{x_n(t,\omega)\}$, $n \in \mathbb{N}$, $t \in [t_0, T]$ may be studied using different approaches, such as converges with probability one (denoted by w. p. 1) to $x(t,\omega)$ if [27]:

$$p\left(\left\{\omega \in \Omega: \lim_{n\to\infty} x_n(t,\omega) = x(t,\omega)\right\}\right) = 1$$

which is also called almost sure convergence.

## 4. The Existence and Uniqueness Theorem

In stochastic fuzzy set theory, consider the ball $B_\rho(\tilde{x}_0) = \{\tilde{u} \in E^d \mid D(\tilde{u}, \tilde{x}_0) \leq \rho\}$, where $\rho > 0$ and let us assume the mapping $f: [t_0, T] \times B_\rho(\tilde{x}_0) \longrightarrow E^d$, which satisfies:

(f1) The mapping $f(t, \tilde{u})$ is a fuzzy random variable, for every $(t, \tilde{u}) \in [t_0, T] \times B_\rho(\tilde{x}_0)$.

(f2) The mapping $f(t, \tilde{u})$ is continuous function with $p.1$.

(f3) There exists a constant $M > 0$, such that:

$$D\big(f(t, \tilde{u}), \tilde{\theta}\big) \overset{[t_0,T]\times B_\rho(\tilde{x}_0).p.1}{\leq} M$$

In a more explanatory way, we can say that $p(\{\omega \in \Omega | \tilde{x}(t,\omega) = \tilde{y}(t,\omega)\}) = 1$, where $\tilde{x}$ and $\tilde{y}$ are fuzzy random elements considered as stochastic processes, we will often denote the equality of $\tilde{x}$ and $\tilde{y}$ with $p.1$ as $\tilde{x}(t,\omega) \overset{p.1}{=} \tilde{y}(t,\omega)$ . Also, if $p(\{\omega \in \Omega \mid \tilde{x}(t,\omega) = \tilde{y}(t,\omega), \forall\, t \in A \subset [t_0, T]\}) = 1$, then it will be abbreviated as $\tilde{x}(t,\omega) \overset{A.p.1}{=} \tilde{y}(t,\omega)$. Similarly, proceeding for the inequalities and other relations, [12].

Now, consider the fuzzy random ordinary differential equation:

$$\left.\begin{aligned} \tilde{x}'(t,\omega) &\overset{[a,b].p.1}{=} f(t, \tilde{x}(t,\omega)) \\ \tilde{x}(t,\omega) &\overset{p.1}{=} x_0 \end{aligned}\right\} \quad \ldots(1)$$

where $t_0 \in \mathbb{R}, x_0 \in E^d, f: [a,b] \times E^d \longrightarrow E^d$ , $\omega$ is the Wiener process and $\tilde{x}(t,\omega)$ is a fuzzy random process.

To state the existence and uniqueness theorem, let $f(t, \tilde{x})$ be continuous function defined on a domain $D \subseteq \mathbb{R}^2$, so that $f$ is Lipschitz continuous fuzzy function with respect to $\tilde{x}$ on $D$ and hence to prove that there exists a unique solution to the initial value problem (1). If:

$$M = \max_{(t,x)\in\mathbb{R}} |f(t, \tilde{x})|$$
$$R = \{(t, \tilde{x}): |t - t_0| \leq a, |\tilde{x} - \tilde{x}_0| \leq b \mid \subset D$$

If $f$ is continuous, then also $f^n$, for all $n \in \mathbb{N}$ are continuous on $D$ and Lipschitz continuous with respect to $\tilde{x}$ on $D$, then there exist a unique solution to the initial value problem on an interval $|t - t_0| \leqslant a$, $a$ and $b$ are parameters of rectangle. Furthermore, the unique solution can be computed form the successive approximation:

$$\tilde{x}_{n+1}(t,\omega) \overset{[a,b].p.1}{=} x_0 + \int_0^t f(s, \tilde{x}_n(t,\omega)ds \quad \ldots(2)$$
$$\tilde{x}_0(t,\omega) = \tilde{x}_0$$

This is known as the Picard's iterations.

The proof of this theorem is based on using Banach contraction mapping principle. Also, we note that in the Banach contraction principle, if $T$ is a contraction, then $T$ has a unique fixed point. If $T^n$ is a contraction for some $n = 1,2,\cdots$, then also $T$ has a unique fixed point and this is known as a generalized Banach contraction mapping principle. So, the generalized Banach contraction principle will be used to establish the proof of the existence and uniqueness theorem of the solution of the fuzzy random initial value problem (1).

**Lemma 4.1, [11].** Let $\tilde{x}: [a,b] \times \Omega \longrightarrow E^n$ be a fuzzy stochastic process, then $\tilde{x}$ is the solution of fuzzy random initial value problem (1) if and only if $\tilde{x}$ satisfies the following fuzzy random integral equation:

$$\tilde{x}(t,\omega) \overset{[a,b].p.1}{=} \tilde{x}_0(\omega) + \int_{t_0}^{t} f(s,\tilde{x}(s,\omega))\, ds \qquad \ldots(3)$$

The integral equation (3) will be written in terms of its lower and upper solutions $\underline{x}$ and $\overline{x}$ based on the $\alpha$-level sets, as follows:

$$\underline{x}(t,\omega) \overset{[a,b].p.1}{=} \underline{x}_0(\omega) + \int_{t_0}^{t} f(s,\underline{x}(s,\omega),\overline{x}(s,\omega))\, ds \qquad \ldots(4)$$

$$\overline{x}(t,\omega) \overset{[a,b].p.1}{=} \overline{x}_0(\omega) + \int_{t_0}^{t} f(s,\underline{x}(s,\omega),\overline{x}(s,\omega))\, ds \qquad \ldots(5)$$

Now, let us define the corresponding operator, so define at first a function as a fuzzy stochastic process $\tilde{x} \epsilon C[a,b]$ with supermum norm defined by $\| \tilde{x} \| = \sup|\tilde{x}(t,\omega)|$ and $(X, \|\cdot\|)$ will be a complete normed space (Banach space). Thus equation (2) can be written in operator form as:

$$\tilde{x}(t,\omega) = \tilde{x}_0 + \int_a^t f(s,\tilde{x}(s,\omega))\, ds$$

and define an operator $T$ by:

$$(T\tilde{x})(t,\omega) \overset{[a,b].p.1}{=} \tilde{x}_0 + \int_a^t f(s,\tilde{x}(s,\omega))ds \qquad \ldots(6)$$

If $T$ has a fixed point, i.e.; there exists $\tilde{x}^* \in X$, such that $T\tilde{x}^* = \tilde{x}^*$, then the fixed point $\tilde{x}^*$ of equation (6) is a solution of the fuzzy random integral equation (2). Therefore, the solution of the fuzzy random integral equation (2) is equivalent to the existence of a fixed point for the fuzzy operator $T$. Therefore, our objective is to show that $T$ has a fixed point, which means the equivalence between the unique solution of the fuzzy RODE (1) and $T$ has a unique fixed point fuzzy the random integral equation (2). We will show that $T^n$ is a contraction for any $n \in \mathbb{N}$. Before we indulge with the proof of the existence and uniqueness, we need to recall the Gronwall's inequality given in the next lemma:

**Lemma 4.2 (Gronwall's inequality), [22].** Suppose that $f$ and $g$ are cautiously real valued functions with $f(x) \geq g(x) \geq 0$ on the interval $[a,b]$. If:

$$f(x) \leq C + K\int_a^x g(s)\, ds,\ C, K \geq 0$$

Then:

$$f(x) \leq Ce^{\int_a^x g(s)\, ds}$$

**Theorem 4.3.** Consider the first order fuzzy random ordinary differential equation:

$$\tilde{x}'(t,\omega) \overset{[a,b].p.1}{=} f(t,\tilde{x}(t,\omega)),\ \tilde{x}(t_0,\omega) \overset{p.1}{=} \tilde{x}_0 \qquad \ldots(7)$$

where $f: f: [a,b] \times E^d \longrightarrow E^d$ and if $f$ satisfies Lipschitz condition with respect to $\tilde{x}$, with constant $K > 0$, such that $K \leq \left[\frac{\alpha^n}{e^{\alpha(b-a)}}\right]^{1/n}$, $n \in \mathbb{N}$. Then the fuzzy random ordinary differential equation (7) has a unique solution as a fuzzy random process.

**Proof.** Let $\tilde{x}_1$ and $\tilde{x}_2$ be two fuzzy random processes and since f satisfies Lipschitz condition, then for $K > 0$:

$$\|f(t,\tilde{x}_1(t,\omega)) - f(t,\tilde{x}_2(t,\omega))\| \leq K\|\tilde{x}_1 - \tilde{x}_2\|$$

The $\alpha$-level intervals of $\tilde{x}_1$ and $\tilde{x}_2$ are $\tilde{x}_1 = [\underline{x}_1, \overline{x}_1]$, $\tilde{x}_2 = [\underline{x}_2, \overline{x}_2]$ and hence to prove the mapping $f$ is a contraction mapping

Since $f$ satisfies Lipschitz condition with respect to $\tilde{x}$, with constant $K > 0$, then in terms of the lower and upper solutions, $f$ will satisfy:

$$\left\|f\left(t,\underline{x}_1(t,\omega),\overline{x}_1(t,\omega)\right) - f\left(t,\underline{x}_2(t,\omega),\overline{x}_2(t,\omega)\right)\right\| \leq K\left[\left\|\underline{x}_1 - \underline{x}_2\right\| + \|\overline{x}_1 - \overline{x}_2\|\right]$$

for all $\underline{x}_1, \overline{x}_1, \underline{x}_2, \overline{x}_2 \in C[a,b]$, and for all $t \in [a,b] \subset \mathbb{R}$, $\tilde{x}_0 \in E^d$, $\omega$ is the Weiner process

Since $\underline{x}_1, \overline{x}_1, \underline{x}_2, \overline{x}_2 \in C[a,b]$, then by equation (6), we have:

$$(T\tilde{x})(t,\omega) \overset{[a,b].p.1}{=} \tilde{x}_0 + \int_a^t f(s,\tilde{x}(s,\omega))ds$$

where $\tilde{x}(t,\omega) = [\underline{x}(t,\omega), \overline{x}(t,\omega)]$.

Also, equation (6) could be written in terms of its lower and upper terms, which are related to the $\alpha$-levels of the fuzzy function $\tilde{x}$, as follows:

$$(T\underline{x})(t,\omega) = \underline{x}_0 + \int_a^t f_s(s,\underline{x}(s,\omega),\bar{x}(s,\omega))\, ds$$

$$(T\bar{x})(t,\omega) = \bar{x}_0 + \int_a^t f_e(s,\underline{x}(s,\omega),\bar{x}(s,\omega))\, ds$$

Thus, the operator integral equations related to $\underline{x}_1$ and $\underline{x}_2$ are:

$$(T\underline{x}_1)(t,\omega) \overset{[a,b].p.1}{=} \underline{x}_0 + \int_a^t f_s(s,\underline{x}_1(s,\omega),\overline{x}_1(s,\omega))\, ds \qquad \ldots(8)$$

$$(T\underline{x}_2)(t,\omega) \overset{[a,b].p.1}{=} \underline{x}_0 + \int_a^t f_s(s,\underline{x}_2(s,\omega),\overline{x}_2(s,\omega))\, ds \qquad \ldots(9)$$

Subtracting equation (9) from equation (8) and taking the supremum norm, getting:

$$\begin{aligned}\left\|\left(T\underline{x}_1\right)(t,\omega) - \left(T\underline{x}_2\right)(t,\omega)\right\| &\overset{[a,b].p.1}{=} \Big\|\underline{x}_0 - \underline{x}_0 + \int_a^t f\left(s,\underline{x}_1(s,\omega),\bar{x}_1(s,\omega)\right)ds - \int_a^t f\left(s,\underline{x}_2(s,\omega),\bar{x}_2(s,\omega)\right)ds\Big\| \\ &\overset{[a,b].p.1}{\leq} \int_a^t \left\|f\left(s,\underline{x}_1(s,\omega),\bar{x}_1(s,\omega)\right) - f\left(s,\underline{x}_2(s,\omega),\bar{x}_2(s,\omega)\right)\right\|ds \\ &\overset{[a,b].p.1}{\leq} K\int_a^t \left[\left\|\underline{x}_2(s,\omega) - \underline{x}_1(s,\omega)\right\| + \|\bar{x}_2(s,\omega) - \bar{x}_1(s,\omega)\|\right] ds\end{aligned}$$

By Gronwall's inequality, getting:

$$\left\|\left(T\underline{x}_1\right)(t,\omega) - \left(T\underline{x}_2\right)(t,\omega)\right\| \overset{[a,b].p.1}{\leq} K\int_a^t \left[\left\|\underline{x}_2(s,\omega) - \underline{x}_1(s,\omega)\right\| e^{\int_a^s \|\bar{x}_2(t,\omega) - \bar{x}_1(t,\omega)\|dt}\right] ds$$

Consider $\|\bar{x}_2(s,\omega) - \bar{x}_1(s,\omega)\| \overset{[a,b].p.1}{=} \alpha$, then:

$$\left\|\left(T\underline{x}_1\right)(t,\omega) - \left(T\underline{x}_2\right)(t,\omega)\right\| \overset{[a,b].p.1}{\leq} K\left\|\underline{x}_2(s,\omega) - \underline{x}_1(s,\omega)\right\| \int_a^t e^{\alpha(s-a)}ds$$

$$\overset{[a,b].p.1}{\leq} K\frac{e^{\alpha(t-a)}}{\alpha}\left[\left\|\underline{x}_1(t,\omega)-\underline{x}_2(t,\omega)\right\|\right]$$

To show that $T^n$ is a contraction if we can bound above equation, which will be less than 1. So, putting some restriction on the Lipschitz constant $K$ by taking the composition. Starting with $T^2$ as follow:

$$\left\|T^2\underline{x}_1(t,\omega)-T^2\underline{x}_2(t,\omega)\right\| \overset{[a,b].p.1}{=} \left\|T\big(T\underline{x}_1(t,\omega)\big)-T\big(T\underline{x}_2(t,\omega)\big)\right\|$$
$$\overset{[a,b].p.1}{\leq} K^2\left(\int_a^t \frac{e^{\alpha(s-a)}}{\alpha}ds\right)\left(\left\|\underline{x}_1(s,\omega)-\underline{x}_2(s,\omega)\right\|\right)$$
$$\overset{[a,b].p.1}{\leq} K^2\frac{e^{\alpha(t-a)}}{\alpha^3}\left\|\big(\underline{x}_1(t,\omega)-\underline{x}_2(t,\omega)\big)\right\|$$

Continue in this manner to take one more composition, say for $T^3\underline{x}_1(t,\omega)-T^3\underline{x}_2(t,\omega)$, implies to:

$$\left\|T^3\underline{x}_1(t,\omega)-T^3\underline{x}_2(t,\omega)\right\| \overset{[a,b].p.1}{\leq} K^3\frac{e^{\alpha(t-a)}}{\alpha^2}\left(\left\|\underline{x}_1(s,\omega)-\underline{x}_2(s,\omega)\right\|\right)$$

and so on for $n$-compositions of the operator $T$, one may get:

$$\left\|T^n\underline{x}_1(t,\omega)-T^n\underline{x}_2(t,\omega)\right\| \overset{[a,b].p.1}{\leq} K^n\frac{e^{\alpha(t-a)}}{\alpha^n}\left(\left\|\underline{x}_1(s,\omega)-\underline{x}_2(s,\omega)\right\|\right)$$
$$\overset{[a,b].p.1}{\leq} K^n\frac{e^{\alpha(b-a)}}{\alpha^n}\left(\left\|\underline{x}_1(s,\omega)-\underline{x}_2(s,\omega)\right\|\right)$$

The value $K^n\frac{e^{\alpha(b-a)}}{\alpha^n}<1$, we must take $K\leq\left[\frac{\alpha^n}{e^{\alpha(b-a)}}\right]^{1/n}$

So $T^n$ is a contraction mapping for $n$ large enough and hence $T$ has unique fixed point as a fuzzy random process. Then by the Banach fixed point theorem. The integral operator $T$ has a unique solution as a fuzzy random process, i.e.; the fuzzy random ordinary differential equation (1) has a unique solution.

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