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On Almost and Star α-Hurewicz Spaces

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Abstract

The main purpose of this work is to create a type of topological spaces namely "almost star α-Hurewicz spaces" and study its properties, and besides, the concepts of α-compact space, α-Hurewicz space, star α-Hurewicz space and strongly star α-Hurewicz space. Many properties of α-Hurewicz spaces and almost α-Hurewicz are investigated. This allows us to provide new examples of explicit descriptions of spaces as well as some types of α-covering for spaces such as α-compact and α-Lindelof space and using as a tool to prove important results in topological spaces. In addition, a certain connection of α-Hurewicz space with the Hurewicz space and almost α-Hurewicz space was considered. There is a relationship between the version of the strongly star α-Hurewicz property and star α-Hurewicz property with star α-compact property and almost star α-Hurewicz. Some of the examples that make a distinction between the properties are mentioned and reviewed, showing that the concepts are not equivalent. Results on the preservation of the properties of α-Hurewicz and almost-α-Hurewicz spaces are included such as behavior under subspaces, products, α-irresolute function, and mappings with other forms of topological spaces.

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1. Introduction

Weak and strong definitions of open sets have been applied by many authors [1-3]. They have given rise to new concepts of continuity: Ec-continuous and δ-ßc-continuous [4], new types of totally continuous[5], faintly θ-semi-continuous, and faintly δ-semi-continuous functions [6]. The generalization of open sets has played important role in many works in concepts like games theory, graph theory and soft topology [7-9]. Also, the concept of generalization of topological spaces used certain types of open sets [10]. Besides, covering properties have been studied in different forms of open sets [11]. In Topological spaces (for short T.s), for a subset $\mathcal{A}$ of a space X, the notations cl($\mathcal{A}$), int($\mathcal{A}$) stand for the closure and the interior of $\mathcal{A}$, respectively. The meaning of $\mathcal{T}_{\mathcal{A}}$ is the topology on $\mathcal{A}$ inherited from a space X with a topology $\mathcal{T}$. The notion of α-open sets was introduced by Njastad [12]; a subset $\mathcal{A}$ of a T.s. X is said to be α-open set, if $\mathcal{A} \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathcal{A})))$ and α-closed if it is the complement of an α-open set. Since the concept of α-open sets has played a role in several significant places in the study of T.s's, the relevance of the definition presented is evidenced by previous studies. A T.s . X is said to be α -compact (respectively, α-Lindelof) space, if for every α-open cover of X , $\{U_j : j \in J\}$, a finite (respectively, a countable) subcover [13] can be found. A T.s. X is called a countably α -compact space, if of each countable set of open α-compact subsets that covers X it is possible to get a finite subcover [14]. A Menger and Hurewicz properties are one of the most important kinds of selection principles. A T.S X has the Menger (resp .Hurewicz) property, if for every sequence $((\mathfrak{U}_n)_{n\in\mathbb{N}}$ of open covers of X there exists a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that every $\mathcal{V}_n$ is a finite subset of $\mathfrak{U}_n$ and the family $\cup\{V: V \in \mathcal{V}_n, n \in \mathbb{N}\}$is a cover of X (resp. each $x \in X$ belongs to $\cup \mathcal{V}_n = \cup\{V: V \in \mathcal{V}_n, n \in \mathbb{N}\}$ for all but finitely many n). The concept of α-open set will be used to define a new form of Hurewicz space. The study in this paper revolves around a new type of T.s, which generalizes the

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Hurewicz property and as a study close to what was presented in previous studies about Hurewicz property. Moreover, " α -Hurewicz property" is discussed, where some of the main characteristics of this space were presented. A subset $\mathcal{A}$ of a T.s X is said to be β-open set, if $\mathcal{A} \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathcal{A})))$ and β-closed the complement of β-open set [14]. A subset $\mathcal{A}$ of T.s X, is said to be a semi-open set (shortly s-open) [15], if $\mathcal{A} \subseteq \mathrm{cl}(\mathrm{int}(\mathcal{A})$. A subset $\mathcal{A}$ of a T.s X is called regular open set if $\mathcal{A} = \mathrm{int}(\mathrm{cl}(\mathcal{A}))$, (respectively, regular closed if $\mathcal{A} = \mathrm{cl}(\mathrm{int}\,(\mathcal{A}))$. Following a natural way, the intersection of all α-closed sets of X containing $\mathcal{A}$ is said to be the α-closure of $\mathcal{A}$, written as $\mathrm{cl}_\alpha(\mathcal{A})$ [12]. The union of all α-open sets of X contain in $\mathcal{A}$ is said to be α-interior of $\mathcal{A}$, written as $\mathrm{int}_\alpha(\mathcal{A})$ [12]. The definition of α-closed subset is equivalent to $\mathcal{A} = \mathrm{cl}_\alpha(\mathcal{A})$. The family of α-open (β-open and s-open, respectively) subsets of X is denoted by $\mathcal{T}^\alpha$ ($\mathcal{T}^\beta$ and $\mathcal{T}^s$ respectively). It is shown that each of $\mathcal{T} \subseteq \mathcal{T}^\alpha$ and $\mathcal{T}^\alpha$ is a topology on X[12]. The collection $\mathcal{T}^\beta$ is not a topology for X because the intersection of β-open sets is not in general a β-open set. Take, for instance, $(\mathbb{R}, \mathcal{T}_u)$, and the intervals (0, 1] and [1, 2]. In the same way of definition $\mathrm{cl}_\alpha(\mathcal{A})$ and $\mathrm{int}_\alpha(\mathcal{A})$, the concept of $\mathrm{cl}_\beta(\mathcal{A})$ ($\mathrm{int}_\beta(\mathcal{A})$), and $\mathrm{cl}_s(\mathcal{A})$ ($\mathrm{int}_s(\mathcal{A})$) was defined, respectively. For any subset $\mathcal{A}$ of X, $\mathrm{int}(\mathcal{A}) \subseteq \mathrm{int}_\beta(\mathcal{A}) \subseteq \mathcal{A} \subseteq \mathrm{cl}_\beta(\mathcal{A}) \subseteq \mathrm{cl}(\mathcal{A})$, $\mathrm{int}(\mathcal{A}) \subseteq \mathrm{int}_s(\mathcal{A}) \subseteq \mathcal{A} \subseteq \mathrm{cl}_s(\mathcal{A}) \subseteq \mathrm{cl}(\mathcal{A})$ and $\mathcal{T} \subseteq \mathcal{T}^\alpha \subseteq \mathcal{T}^s \subseteq \mathcal{T}^\beta$. In addition, the properties of α-Hurewicz as an image or preimage of special types of continuous mappings are studied. Newly, the concept of α-covering property have been examined with a variation, after applying the interior and the closure operators on a Hurewicz property [16]. Furthermore, different forms have been studied in case of the sequence of open covers are changed with generalized open sets [16]. In connection with this notion, the Menger property is very similar to the Hurewicz property although, analyzed in locales in [17], it is a stronger condition,

2. α-Hurewicz Spaces

This section deals with the statement of results about α -Hurewicz spaces and besides, some examples of topological spaces are provided to show the relationships among Hurewicz, α-Hurewicz, β-Hurewicz, s-Hurewicz spaces and another types of spaces such that α-compact and α-Lindelof spaces.

Definition 2.1. [16] Let X be a $T.s$ and $\mathcal{A} \subseteq X$. Then $\mathcal{A}$ has α-Hurewicz property, if ∀ sequence $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ of α-open covers of $\mathcal{A}$, ∃ sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ for any $n \in \mathbb{N}$,where $\mathcal{V}_n$ is a finite subset of $\mathfrak{U}_n$. Also for every $x \in \mathcal{A}$ satisfied that $x \in \bigcup \mathcal{V}_n$ for all but finitely many n,. A T. s X is α-Hurewicz space when the set X is α-Hurewicz.

Example 2.1. Take $X = \mathbb{Z}^+$ (positive integers) with $\mathcal{T}_{dis}$ (discrete topology). So, $\mathcal{T}_{dis} = \mathcal{T}^\alpha$ and hence $(X, \mathcal{T}_{dis})$ is α-Hurewicz space.

The following examples show the relation between a compact space (respectively, Lindelof Hurewicz, α-compact, α -Lindelof) and α -Hurewicz with the following corresponding spaces in ($\mathcal{T}^s$ and $\mathcal{T}^\beta$ respectively). Some concepts are recalled in Definition 2.2.

Definition 2.2. A topological space$(X, \mathcal{T})$ is said to be
(i) *semi* -Hurewicz[13](resp. $\beta - Hurewicz$ [18]) if for every sequence $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ of semi open (resp. β-open) cover there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ for any $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathfrak{U}_n$ and for each $x \in X$ for all but finitely many n, with $x \in \bigcup \mathcal{V}_n$.
(ii) α- Lindelof [18] if for all cover $\{\mathcal{A}_j \mid j \in \mathrm{J}\}$ of X, being $\mathcal{A}_j$ $(j \in J)$ $\alpha -$ open sets, there is a countable sub cover.

Evidently, the following implications are hold:

β -Hurewicz ⇒ s -Hurewicz ⇒ α -Hurewicz ⇒ Hurewicz

It is simple to show that every α-compact space is α-Hurewicz space, but the converse does not necessarily hold, for instance, let $\mathrm{X} = \mathbb{Z}$ with $\mathcal{T} = \mathcal{T}_{\mathrm{dis}}$. Then X is α-Hurewicz space, but it is not α-compact, since $\{\{x\} : x \in X\}$ be α-open cover of X has no a finite subcover.

Also, every α-Hurewicz space is Hurewicz space, but the converse is not true as the following example. Let A be a finite subset of an uncountable set X. Then $\mathcal{T} = \{\emptyset, \mathrm{A}, \mathrm{X}\}$ is a topology on X. The space $(\mathrm{X}, \mathcal{T})$ is Hurewicz but it is not an α-Hurewicz space because the sequence of an α-open cover $\mathfrak{U}_n = \{\mathrm{A} \cup \{x\}: x \in \mathrm{X} \setminus \mathrm{A}\}$ for each $\mathrm{n} \in \mathbb{N}$, because it is not possible to find a countable subcover of the cover $\mathfrak{U}_n$.

It is easily established that if X is a s-Hurewicz space, then X is an α-Hurewicz space, however, the converse does not necessarily hold. Indeed, let $\mathrm{X} = \mathbb{R} \cup \wp$, where $\wp$ be a countable set and $\mathbb{R} \cap \wp = \emptyset$ with a topology, which is defined by

$$\mathcal{T} = \{\mathcal{U} \subseteq \wp : \mathcal{U}^c \text{is a finite subset of } \wp\} \cup \mathcal{T}_u.$$

Here, X is a α-Hurewicz space, but it is not s-Hurewicz space. Additionally, every $\beta -$ Hurewicz space (respectively semi -Hurewicz) is α-Hurewicz

space but the converse is not true as it happens considering $X = \mathbb{R}$, with $\mathcal{T} = \mathcal{T}_{ind}$ (indiscrete topology). Here, the topological space is α-Hurewicz (s-Hurewicz respectively), but not is β-Hurewicz.
Moreover, if X is s -Hurewicz (respectively α-Hurewicz) then it is Hurewicz but, the converse is not satisfied in the next example. Let $X = \mathbb{R}$, with a usual metric topology $\mathcal{T}_u$. Here, X is a Hurewicz space. In the proof is essential the fact of $[-n, n]$ is compact. Nevertheless, it is not s-Hurewicz space, since $\mathfrak{U}_n = \{[r,\ r + \frac{m-1}{m}), r \in \mathbb{Z}, m \in \mathbb{N}\}$ is a sequence of cover of X, ($[r,\ r + \frac{m-1}{m})$, is s-open and $[r, r+1]$ is not s-compact), and it is not possible to find a finite subfamily of each $\mathfrak{U}_n$ such that $\mathbb{R}$ is covered by the union. As an example of α-Hurewicz (α-Lindelof respectively) space take the set $X = [0, \omega_1]$, with the ordinal topology, while if the set $X = [0, \omega_1)$ is taken, with the ordinal topology, is not α-Hurewicz (is not α -Lindelof) space. The family $\{\mathcal{U}_\alpha = [0, \alpha) : \alpha \in [0, \omega_1)\}$ is an α -open cover of $[0, \omega_1)$ with no countable subcover.

Gaurav et al. proved in [16] that α -Hurewicz property is not hereditary property, and study α-continuity of α-Hurewicz spaces. Thus, the below example and results can be established.

Example 2.2. Suppose that $X = \mathbb{R}$, define a basis $\mathcal{B} = \{\mathcal{U} : \mathcal{U} \subseteq \mathbb{R}\}$; for a topology $\mathcal{T}$ on X, with $\mathcal{U} = \begin{cases} \{r\} & ; r \in X \setminus \{0\} \\ 0 \in \mathcal{U} ; \mathcal{U}^c \text{ countable} \end{cases}$. It is clear that X is α-Hurewicz space.
Let us take $Y = \{\{r\} : r \in X \setminus \{0\}\}$ is a subspace of X. As for any sequence of α-open covers of Y has no countable subcover, then Y is not α-Hurewicz space.
The following proposition is proved with regular closed condition, and we do not need the clopen (i.e., closed and open) condition as in [16].

Proposition 2.1. Let $(X, \mathcal{T})$ be the α-Hurewicz space and $Y \subseteq X$. If Y is a regular closed set of X, then Y has the α-Hurewicz property.

Proof. Consider Y a regular closed subspace of the α-Hurewicz space X and $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ a sequence of α-open covers of Y. Let $\mathfrak{G}_n = \{\mathcal{U} : \mathcal{U} \in \mathfrak{U}_n\} \cup \{X \setminus Y\}$, $n \in \mathbb{N}$. As Y is closed then $X \setminus Y$ is open and so α-open. Hence $(\mathfrak{G}_n)_{n\in\mathbb{N}}$ is a sequence of α-open covers of X. By the α-Hurewiczness property of X, it is possible to obtain a sequence $(\mathcal{W}_n)_{n\in\mathbb{N}}$ with $\mathcal{W}_n$ is a finite subset of $\mathfrak{G}_n$ for each $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{W}_n$. Taking for each n, $\mathcal{V}_n = \{\mathcal{U} : \mathcal{U} \in \mathcal{W}_n\}$, the sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ is a finite subset of $\mathfrak{U}_n$ and each $x \in Y$ for all but finitely many n, with $x \in \bigcup \mathcal{V}_n$. That is Y has the α-Hurewicz property. The following theorem states that the α-Hurewiczness is presented under α − irresolute mapping.

Definition 2.3. [19] Let $g : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ be a function between to topological spaces, then g is α-irresolute if the inverse image of α- open is α- open.

Remark 2.1. A subspace of a product of spaces does not need to be α-Hurewicz and neither is the product space as the next example shows.

Example 2.3. Consider the Sorgenfrey line $\mathbb{S}$, i.e., the set $\mathbb{R}$ endowed by the topology provided by the base $\mathcal{B} = \{[x, y) : x < y, x, y \in \mathbb{R}\}$. Then for any oblique line with negative slope $L = \{(r, s) \in \mathbb{S} \times \mathbb{S} : s = ar + b, a < 0\}$ endowed by $\mathcal{T}_L$, the inherited topology of $\mathbb{S} \times \mathbb{S}$. L is not α-Hurewicz because of $\mathcal{T}_L = \mathcal{T}_{dis}$ and neither does $\mathbb{S} \times \mathbb{S}$.
Assume that $\mathbb{S} \times \mathbb{S}$ is α-Hurewicz. The proof is based on the fact: every α-Hurewicz is α-Lindelof. Let us take $L \subseteq \mathbb{S} \times \mathbb{S}$. It is uncountable, as its cardinal is the same as the cardinal of $\mathbb{R}$. From α-closedness of L in $\mathbb{S} \times \mathbb{S}$, implies that $\mathbb{S} \times \mathbb{S}$ is not α-Lindelof, which is contradicts α -Hurewiczness of $\mathbb{S} \times \mathbb{S}$. Consequently, L is α-closed by $(\mathbb{S} \times \mathbb{S}) \setminus L$ is α-open in $\mathbb{S} \times \mathbb{S}$. Indeed, let $L^+ = \{(r, s) : s - ar - b > 0\}$ and $L^- = \{(r, s) : s - ar - b < 0\}$. So, $(\mathbb{S} \times \mathbb{S}) \setminus L = L^+ \cup L^-$. Let $(r, s) \in L^+$. So, every α-open set contains (r, s) intersects more than one point with L (since we can write it as $[r, r + \epsilon) \times [s, s + \epsilon)$. But $[r, \frac{-ar-b+s}{2}) \times [s, \frac{-ar-b+s}{2})$ does not intersect L. If $(r, s) \in L^-$, then the α-nbhd $[r, \frac{ar+s-b}{2a}) \times [s, \frac{ar+b+3s}{4})$ does not intersect L. The sets L and L^- are both α-open in $\mathbb{S} \times \mathbb{S}$, hence, L is α-closed. Now, for every $(r, s) \in L$, each α-nbhd of (r, s) in $\mathbb{S} \times \mathbb{S}$ (it can be written by $[r, r + \epsilon) \times [s, s + \epsilon)$, for $\epsilon > 0$) intersects L in just one point, (r, s). Therefore, the property is proven.

Theorem 2.1. The product of an α-Hurewicz space and an α-compact space is α-Hurewicz.

Proof: Fix X an α -Hurewicz space and Y an α-compact space. To show that $X \times Y$ is α-Hurewicz space, consider $(\mathcal{W}_n)_{n\in\mathbb{N}}$ a sequence of α-open covers of $X \times Y$. Hence, there exists α-open covers $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ and $(\mathcal{V}_n)_{n\in\mathbb{N}}$ of X and Y, respectively such that $\mathcal{W}_n = \mathfrak{U}_n \times \mathcal{V}_n$. By α-Hurewiczness of X, a sequence $(\mathfrak{U}'_n)_{n\in\mathbb{N}}$ can be taken with $\mathfrak{U}'_n$ are finite subsets of $\mathfrak{U}_n$ for each $n \in \mathbb{N}$ and for each $x \in X$ for all but finitely many n, with $x \in \bigcup \mathfrak{U}'_n$. Also, from α-compactness of

Y, choose a finite subset $\mathcal{V}'_n$ of $(\mathcal{V}_n)_{n\in\mathbb{N}}$ which is α-open covers of Y. Now, consider $\mathcal{P}_n = \mathfrak{U}'_n \times \mathcal{V}'_n$. Hence for each $n \in \mathbb{N}$, $\mathcal{P}_n$ is a finite subset of $\mathcal{W}_n$ and for each $(x, y) \in X \times Y$ for all but finitely many n, with $(x, y) \in \bigcup \mathcal{P}_n$, which concludes the proof.

Remark 2.2. Recall that in a $T.s$ X and let $\mathfrak{A}$ be a collection of subsets of X. If $\mathcal{A}$ is a subset of X, then the star of $\mathcal{A}$ with respect to $\mathfrak{A}$, denoted by $St(\mathcal{A}، \mathfrak{A})$, is the set $\{\mathfrak{U} \in \mathfrak{A} : \mathfrak{U} \cap \mathcal{A} \neq \emptyset\}$; for $\mathcal{A} = \{x\}$ such that $x \in X$, $St(x، \mathfrak{A})$ is written instead of $St(\{x\}، \mathfrak{A})$.

Definition 2.4. The space X is called star α-Hurewicz space, if for any sequence $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ of α-open covers of X, sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ can be obtained for any $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathfrak{U}_n$ and for each $x \in X$, $x \in St(\bigcup \mathcal{V}_n ، \mathfrak{U}_n)$ for all but finitely many n.
As an example of star α-Hurewicz space, take $X = \mathbb{Z}^+$ the set of positive integers with $\mathcal{T}_{dis}$ (discrete topology). So, X is star α-Hurewicz space.

Definition 2.5. The space X is called strongly star α-Hurewicz space, if for any sequence $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ of α-open covers of X, a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of finite subsets of X can be obtained, for any $n \in \mathbb{N}$, $x \in X$, implies that $x \in St(\mathcal{F}_n ، \mathfrak{U}_n)$ for all but finitely many n .
As an example of strongly star α-Hurewicz space, take $\mathbb{Z}^+$ (positive integers) with the topology $\mathcal{T} = \{\mathcal{U} \subseteq X : \mathcal{U} = \{h \in \mathbb{Z}^+ : 0 \leq h \leq n ; n \in \mathbb{Z}^+\}\} \cup \{\emptyset\}$. Thus, $(X, \mathcal{T})$ is a strongly star α-Hurewicz space.

Definition 2.6. The space X *is* called star α-compact space, if for each α-open covering $\mathfrak{A}$ of X, a finite set $\mathcal{A} \subseteq X$ can be obtained such that $St(x_1 ، \mathfrak{A}) = X$.
As an example of star α-compact space. Let $X = \mathbb{Z}^+$ (positive integers) with $\mathcal{T}_{dis}$. So, X is star α-compact space. The star α-compactness is not hereditary property as in the case of the following space. Let X be an arbitrary infinite set, $x_0 \in X$. Define a topology on X as follows: $\mathcal{T} = \{\mathcal{U} \subseteq X : x_0 \notin \mathcal{U}\} \cup \{\mathcal{U} \subseteq X : X \backslash \mathcal{U}$ is finite set$\}$. The subsets $\{x\}$, $x \in X \setminus \{x_0\}$ are α-open. If $\mathfrak{A}$ is an α-open covering of X, there exists $\mathcal{U} \in \mathfrak{A}$ such that $x_0 \in \mathcal{U}$, so $\mathcal{U} = X \setminus \{x_1, \cdots, x_n\}$. Then, it is enough to take $\mathcal{A} = \{x_0, x_1, \cdots, x_n\}$. Hence, X is star α-compact space. However, the subspace $Y = X \setminus \{x_0\}$ is not. Fix the α-open cover $\mathfrak{A} = \{\{x\} : x \in Y\}$ of Y which does not have a countable subcover, therefore Y cannot be star α-compact space. There is a relation among the different shades ofα-Hurewicz spaces as contained in the following proposition.

Proposition 2.2. Let X be a $T.s$. The following statements are holds:

i. Every α-Hurewicz space is star α-Hurewicz space.
ii. Every strongly star α-Hurewicz space is star α-Hurewicz space.
iii. Every star α-compact space is star α-Hurewicz space.

Proof.

i. Consider X an α-Hurewicz space and. $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ any sequence of α-open covers of X. So, a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ can be obtained for any $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathfrak{U}_n$ and for each $x \in X$, for all but finitely many n, with $x \in \bigcup \mathcal{V}_n$. That is, $\bigcup \mathcal{V}_n \cap \mathfrak{U}_n \neq \emptyset$ for all but finitely many n, and hence $x \in St(\bigcup \mathcal{V}_n ، \mathfrak{U}_n)$ for all but finitely many n. Therefore, X is a star α-Hurewicz space.
ii. Let X be a strongly star α-Hurewicz space and take $\mathfrak{A}$ a cover of α-open sets of X. For the constant sequence of open covers $(\mathfrak{U}_n)_{n\in\mathbb{N}}$, where for each n, $\mathfrak{U}_n = \mathfrak{U}$, $\mathfrak{U} \in \mathfrak{A}$ there is a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ such that for n, $St(\mathcal{F}_n ، \mathfrak{U}_n) \in \mathfrak{U}$ (respectively, $St(\bigcup \mathcal{V}_n ، \mathfrak{U}_n) \in \mathfrak{A}$). That is, $St(\mathcal{F}_n ، \mathfrak{U}_n)$ is a countable subset of X with $St(\bigcup \mathcal{F}_n ، \mathfrak{U}) = X$. Consequently, $St(\bigcup \mathcal{V}_n ، \mathfrak{U}_n)$ is a countable subset of $\mathfrak{U}$ such that $St(\bigcup \mathcal{V}_n ، \mathfrak{U}))$. Then, there exists a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ for any $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathfrak{U}_n$ and for each $x \in X$, $x \in St(\bigcup \mathcal{V}_n ، \mathfrak{U}_n)$ for all but finitely many n. Hence X is star α-Hurewicz space.
iii. Suppose that X is star α-compact space and consider $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ a sequence of α-open covers of X. From star α-compactness of X, a finite set $\mathcal{A} \subseteq X$ is found such that $St(\mathcal{A}، \mathfrak{A}) = X$. Therefore, there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ for any $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathfrak{U}_n$ and for each $x \in X$, $x \in St(\bigcup \mathcal{V}_n ، \mathfrak{U}_n)$ for all but finitely many n. Hence X star α-Hurewicz space.

3. Almost and Star α-Hurewicz Spaces

In this section, the concept of almost α-Hurewicz property is introduced and also several examples are included to point the relationships among Hurewicz, α-Hurewicz, β-Hurewicz, s-Hurewicz spaces and another types of spaces such that α-compact and α-Lindelof spaces.

Definition 3.1. Let X be a $T.s.$ and $\mathcal{A} \subseteq X$. Then $\mathcal{A}$ has the almost α-Hurewicz property, if for any sequence $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ of α-open cover of $\mathcal{A}$, where $\mathfrak{U}_n = \{\mathfrak{U}_{n_j}\}_{j\in J_n}$, where $(J_n)_{n\in\mathbb{N}}$ is a sequence of index sets, I_n a finite set, and a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ can be obtained such that:

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i. for any $n \in \mathbb{N}$, there is $I_n \subseteq J_n$; $\mathcal{V}_n = \{cl(\mathfrak{U}_{n_j})\}_{j \in I_n}$.
ii. for each $x \in \mathcal{A}$, there is $n_0 \in \mathbb{N}$; for all $n \in \mathbb{N}$, $n > n_0$ implies that there is $\mathcal{V} \in \mathcal{V}_n$ with, $x \in \mathcal{V}$.

X is called an almost α-Hurewicz space when in the set X is satisfied the almost α-Hurewicz property. *As almost α -Hurewicz space there are examples such that the following.*

Let $X = \mathbb{R}$, with $\mathcal{T} = \mathcal{T}_{ind}$ (indiscrete topology), then components of α -open covers whose singleton elements are transpositions are entirely determined by an almost α -Hurewicz property. Another characterization of almost α-Hurewicz space is given in the next result.

Theorem 3.1. For a space X the condition almost α-Hurewicz space is equivalent to that for each sequence $(\mathfrak{U}_n)_{n \in \mathbb{N}}$ of covers of X by regular open sets, there exists a sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$, such that

i. for any $n \in \mathbb{N}$, there is $I_n \subseteq J_n$; $\mathcal{V}_n = \{cl(\mathfrak{U}_{n_j})\}_{j \in I_n}$.
ii. for each $x \in A$, there is $n_0 \in \mathbb{N}$; for all $n \in \mathbb{N}$, $n > n_0$ implies that there is $\mathcal{V} \in \mathcal{V}_n$ with, $x \in \mathcal{V}$.

Proof: ($\Longrightarrow$) It is obvious, since every regular open set is open.

($\Longleftarrow$) Let $(\mathfrak{U}_n)_{n \in \mathbb{N}}$ be a sequence of $\alpha-$ open set cover of X, such that $\mathfrak{U}_n = \{\mathfrak{U}_{n_j}\}_{j \in J_n}$. Let $\mathfrak{U}'_n = \{int\,(cl(int(\mathfrak{U}_{n_j})))\}_{j \in J_n}$, then $\mathfrak{U}'_n$ is a regular open cover of X, by hypothecs a sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$, is obtained such that

for any $n \in \mathbb{N}$, there is $I_n \subseteq J_n$; $\mathcal{V}_n = \{cl(\mathfrak{U}'_{nj})\}_{j \in I_n}$.
for each $x \in X$, there is $n_0 \in \mathbb{N}$ such that $n \in \mathbb{N}$, $n > n_0$ implies that there is $\mathcal{V} \in \mathcal{V}_n$ with, $x \in \mathcal{V}$.
Since $\mathfrak{U}_{n_j}$ $\alpha-$ open it is followed that $cl(\mathfrak{U}'_{nj}) = cl((\mathfrak{U}_{n_j})$, and hence each $\mathcal{V}_n = \{cl\,(\mathfrak{U}_{n_j})\}_{j \in I_n}$.

Theorem 3.2. If X is an α-Hurewicz space, then X is an almost α-Hurewicz space.

Proof: Fix $(\mathfrak{U}_n)_{n \in \mathbb{N}}$ a sequence of α-open covers of X, $\mathfrak{U}_n = (\mathfrak{U}_{n_j})_{j \in J_n}$. From α-Hurewiczness of X, a sequence $(\mathcal{W}_n)_{n \in \mathbb{N}}$ is obtained such that:
(i) Each $\mathcal{W}_n \subseteq \mathfrak{U}_n$.
(ii) For all $x \in X$, there is $n_0 \in \mathbb{N}$ for all $n \in \mathbb{N}$, $n > n_0$, $I_n \subseteq J_n$.
By (i), it is possible to write $\mathcal{W}_n = (\mathfrak{U}_{n_j})_{j \in I_n}$, where $I_n \subseteq J_n$. Let $(\mathcal{V}_n)_{n \in \mathbb{N}}$ be a sequence of α-open sets defined by $\mathcal{V}_n = (cl(\mathfrak{U}_{n_j}))_{j \in J_n}$. If $\in X$, then by (ii), there is $n_0 \in \mathbb{N}$ such that:

$$\forall\, n \in \mathbb{N}, > n_0 \,,\ \mathcal{W} \in \mathcal{W}_n \text{ with } x \in \mathcal{W}.$$

Since $\mathcal{W} \in \mathcal{W}_n$, $\mathcal{W} = \mathfrak{U}_{n_j}$ for some $j \in I_n$.can be said.
Let $\mathcal{V} = (cl(\mathfrak{U}_{n_j}))_{j \in J_n}$. Then $\mathcal{V} \in \mathcal{V}_n$ and $x \in \mathcal{V}$ since $\mathcal{W} \subseteq \mathcal{V}$. Hence X is almost α-Hurewicz space.

Remark 3.1. The below example indicates that in general the converse of theorem 4 is false.

Example 3.1. Consider X the Euclidean plane endowed with a topology $\mathcal{T}^{DR}$ generated by the base formed by the following sets:
$DR_r(x_0, y_0) = (D_r(x_0, y_0) \setminus \{(x, y) \in D_r(x_0, y_0): x = x_0\}) \cup \{(x_0, y_0)\}$, where $D_r(x_0, y_0)$ is the disk centered in (x_0, y_0) and radius $r > 0$.

This topology is well known as deleted radius topology. As X is not an α-Lindelof space, then X does not verify the α-Hurewicz property.
However, X is almost α -Hurewicz. Indeed, every $DR_r(x_0, y_0)$ is an α -open set and $cl(DR_r(x_0, y_0)) = cl(D_r(x_0, y_0))$. Applying that $\mathbb{R}^2$ with the usual topology is σ-compact it is obtained the almost α-Hurewicz property.

It is concluded the same with X the Euclidean plane endowed with a topology $\mathcal{T}^{BT}$ generated by the base formed by the following sets:
$BT_r(x_0, y_0) = \{(x, y): |y - y_0| < |x - x_0| < r\} \cup \{(x_0, y_0)\}$, $r > 0$. This topology is well known as deleted bow tie topology. Here, $cl\,(BT_r(x_0, y_0))$ is a compact set in the Euclidean plane with the usual topology, too.
Let X be a T.s. the following notions were introduced in:

- X is α -regular [18], if for any $x \in X$ and a closed subset $B \subseteq X$ such that $x \notin B$ there are two disjoint open sets $H_1, H_2 \subseteq X$ such that $x \in H_1$ and $cl(B \cap H_2) = B$.
- X almost α-regular [20], if for any $x \in X$ and a regularly closed subset $B \subseteq X$ such that $x \notin B$ there are two disjoint α- open sets $H_1, H_2 \subseteq X$ such that $x \in H_1$ and $cl(B \cap H_2) = B$.

Theorem 3.3. If X is an almost α-regular space and an almost α -Hurewicz space, then X is an α -Hurewicz space.

Proof: Consider $(\mathfrak{U}_n)_{n \in \mathbb{N}}$ a sequence of α-open covers of X. From almost α-regularness of X, there is for each n an α-open cover $(\mathcal{V}_n)_{n \in \mathbb{N}}$ of X such that $\mathcal{V}'_n = \{cl(\mathcal{V}) : \mathcal{V} \in \mathcal{V}_n\}$ is a refinement of $\mathfrak{U}_n$. By applying the hypothesis, a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ is found such that for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and $\cup\{\mathcal{W}'_n : n \in \mathbb{N}\}$ is α -open cover of X , where $\mathcal{W}'_n =$

$\{cl(\mathcal{W}) : \mathcal{W} \in \mathcal{W}_n\}$. For every $n \in \mathbb{N}$ and every $\mathcal{W} \in \mathcal{W}_n$, choose $\mathfrak{U}_{\mathcal{W}} \in \mathfrak{U}_n$ such that $(\mathcal{W}) \subseteq \mathfrak{U}_{\mathcal{W}}$. Put $\mathfrak{U}'_n = \{cl(\mathcal{V}) : \mathfrak{U}_{\mathcal{W}} \in \mathcal{W}_n\}$. Now, it is shown that $\cup\{\mathfrak{U}'_n : n \in \mathbb{N}\}$ is α-open cover of X. Let $x \in X$. There is $n \in \mathbb{N}$ and $cl(\mathcal{W}) \in \mathcal{W}'_n$ such that $x \in \mathcal{W}$. So, there is $\mathfrak{U}_{\mathcal{W}} \in \mathfrak{U}'_n$ such that $\mathcal{W} \subseteq \mathfrak{U}_{\mathcal{W}}$. Then, $x \in \mathfrak{U}_{\mathcal{W}}$.
Recall that a function $f: X \longrightarrow Y$ is said to be almost α-continuous, if for each regular open set $\subseteq Y$, $f^{-1}(B)$ is an α-open set in X.

Theorem 3.4. If X is an almost α-Hurewicz space, Y is any T.s., and $f: X \longrightarrow Y$ is an almost α-continuous surjection, then Y is an almost α-Hurewicz space.

Proof: By Theorem 3.1 it is sufficient to do the proof for $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ a sequence of α-open covers of Y by α-regular open sets. Assume that $\mathfrak{U}'_n = \{f^{-1}(\mathfrak{U}): \mathfrak{U} \in (\mathfrak{U}_n)_{n\in\mathbb{N}}\}$ for each $n \in \mathbb{N}$. Thus $(\mathfrak{U}'_n)_{n\in\mathbb{N}}$ is a sequence of α-open covers of X, because of f is an almost α-continuous surjection. From almost α-Hurewiczness of X, a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ of X can be found such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathfrak{U}'_n$ and $\cup\{\mathcal{V}'_n : n \in \mathbb{N}\}$ is a α-open cover of X, where $\mathcal{V}'_n = \{cl(\mathcal{V}) : \mathcal{V} \in \mathcal{V}_n\}$. For each $n \in \mathbb{N}$ and $\mathcal{V} \in \mathcal{V}_n$, choose $\mathfrak{U}_{\mathcal{V}} \in \mathfrak{U}_n$ such that $\mathcal{V} = f^{-1}(\mathfrak{U}_{\mathcal{V}})$. Let $\mathcal{W}_n = \{cl(\mathfrak{U}_{\mathcal{V}}): \mathcal{V} \in \mathcal{V}_n\}$. It is only necessary to prove that $\cup\{\mathcal{W}_n : n \in \mathbb{N}\}$ is a cover for X. Now, if $y = f(x) \in Y$, then it is obtained $n \in \mathbb{N}$ and $\mathcal{V}' \in \mathcal{V}'_n$ such that $x \in \mathcal{V}'$. Since $\mathcal{V} = f^{-1}(\mathfrak{U}_{\mathcal{V}})$, $f^{-1}(cl(\mathfrak{U}_{\mathcal{V}}))$ is α− closed, $f(x) \in f(cl(f^{-1}(\mathfrak{U}_{\mathcal{V}})) \subseteq cl(\mathfrak{U}_{\mathcal{V}})$.) Hence, $y = f(x) \in \mathcal{W}_n$.

Definition 3.2. The space X is called almost star α-Hurewicz space, if for any sequence $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ of α-open covers of X, a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ can be obtained for any $n \in \mathbb{N}$ and $x \in X$, $x \in cl(St(\mathcal{V}_n, \mathfrak{U}_n))$ for all but finitely many n.
As an example of almost star α-Hurewicz space, take $X = \mathbb{Z}^+$, with $\mathcal{T}_{dis}$. Then X is almost star α-Hurewicz space.

Theorem 3.5. For X a T.s, the condition almost star α-Hurewicz space is equivalent to that for each sequence $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ of α-open covers of X by of α-regular open sets there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathfrak{U}_n$ and $\{cl(St(\mathcal{V}_n, \mathfrak{U}_n)) : n \in \mathbb{N}\}$ is a cover of X.

Proof: Suppose the condition is fulfilled, then it is clear every α-regular open set is α-open. Conversely, take $(\mathfrak{U}_n)_{n\in\mathbb{N}}$ a sequence of α-open covers of X. Let $\mathfrak{U}'_n = \{int(cl(\mathfrak{U})) : \mathfrak{U} \in \mathfrak{U}_n\}$. So, each $\mathfrak{U}'_n$ covers X by α-regular open sets. Certainly, as since $\mathfrak{U}$ is an α-open set then each $int(cl(\mathfrak{U}))$ is a regular α-open set and $\mathfrak{U} \subseteq int(cl(\mathfrak{U}))$. So, it is possible to find a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathfrak{U}'_n$ and $\{cl(St(\mathcal{V}_n, \mathfrak{U}'_n)) : n \in \mathbb{N}\}$ covers X. Therefore, it is enough to show.

$St(\mathfrak{U}, \mathfrak{U}_n) = St(int(cl(\mathfrak{U})), \mathfrak{U}_n)$ for each $\mathfrak{U} \in \mathfrak{U}_n$.

Now, since $\mathfrak{U} \subseteq int(cl(\mathfrak{U}))$, it is obvious that $St(\mathfrak{U}, \mathfrak{U}_n) \subseteq St(int(cl(\mathfrak{U})), \mathfrak{U}_n)$. Suppose that $x \in St(int(cl(\mathfrak{U}), \mathfrak{U}_n)$. Then there exists $\mathcal{V} \in \mathfrak{U}_n$ such that $x \in \mathcal{V}$ and $\mathcal{V} \cap int(cl(\mathfrak{U}) \neq \emptyset$. So, it is obtained $\mathcal{V} \cap \mathfrak{U} \neq \emptyset$ which implies $x \in St(\mathfrak{U}, \mathfrak{U}_n)$. For every $\mathcal{V} \in \mathcal{V}_n$, choose $\mathfrak{U}_{\mathcal{V}} \in \mathfrak{U}_n$ such that $\mathcal{V} = int(\mathfrak{U}_{\mathcal{V}})$. Let $\mathcal{W}_n = \{\mathfrak{U}_{\mathcal{V}} : \mathcal{V} \in \mathcal{V}_n\}$. Now, it is proved that $cl\{\cup St(\mathcal{W}_n, \mathfrak{U}_n) : n \in \mathbb{N}\}$ is a cover of X. For that, consider $x \in X$. Then it is possible to find $n \in \mathbb{N}$ such that $x \in cl\{St(\cup\mathcal{V}_n, \mathfrak{U}'_n)\}$. For every neighborhood $\mathcal{V}$ of x, $\mathcal{V} \cap St(\cup\mathcal{V}_n, \mathfrak{U}'_n) \neq \emptyset$, then there exists $\mathfrak{U} \in \mathfrak{U}_n$ such that $(\mathcal{V} \cap int(cl(\mathfrak{U})) \neq \emptyset) \wedge (\cup\mathcal{V}_n \cap int(cl(\mathfrak{U}) \neq \emptyset) \neq \emptyset$, implies that $(\mathcal{V} \cap \mathfrak{U} \neq \emptyset) \wedge (\cup\mathcal{V}_n \cap \mathfrak{U}) \neq \emptyset$ then $\mathcal{W}_n \cap \mathfrak{U}_n \neq \emptyset$, so $x \in cl\{\cup St(\mathcal{W}_n, \mathfrak{U}_n)$.

4. Conclusions

Several topics related to the concepts of α-Hurewicz spaces have been treated. Even though α-Hurewicz condition is stronger than Hurewicz condition, in most results quite similar techniques for their proofs work with some adaptations, and thus, α-covering properties of α-Hurewicz have been analyzed. The examples provided show that the property α-Hurewicz property is different from the Hurewicz property and also from the almost α-Hurewicz property (for example see theorems 2.1 and 3.3) As a prospective, these problems for the α-Menger properties, (considering Menger and almost Menger properties) could be studied, so far, as the authors know, they are still open.

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