The Continuous Classical Optimal Control for a Coupled Nonlinear Parabolic Partial Differential Equations with Equality and Inequality Constraints

[1]Jamil A. Ali Al-Hawasy and Ghufran M. Kadhem
Department of Mathematics, College of Science, Al-Mustansiriyah University.
[1]E-mail: hawasy20@yahoo.com.

Abstract

This paper is concerned with the existence and uniqueness state vector solution for a coupled of nonlinear parabolic equations using the Galerkin method when the continuous classical control vector is given, the existence theorem of a continuous classical optimal control vector with equality and inequality vector state constraints is proved, the existence and uniqueness solution of the adjoint equations associated with the state equations is studied. The derivation of the Frcéhet derivative of the Hamiltonian is obtained. Finally the necessary conditions theorem, so as the sufficient conditions theorem of optimality of the constrained problem are proved.

Keywords**:** Classical optimal control, system of nonlinear parabolic, necessary conditions.

1 Introduction

The optimal control problems play an important role in the many fileds in life problems, for examples in robotics [Rubio et al 2011], in an electric power [Aderinto & Bamigbola 2012], in civil engineering [Amini & Afshar 2008], in Aeronautics and Astronautics [Budigono& Wibowo 2007], in medicine [El hiaet al 2012], in economic [Boucekkine& Fabbri 2013], in heat conduction [Borzabadi et al 2004], in biology [Agusto & Bamigbola 2007] and many others fields.

The importance of optimal control problems encourage many researchers interested to study the optimal control problems for systems are governed either by nonlinear ordinary differential equations as in [Warga, 1972] and in [Orpel 2009] or by linear partial differential equations as in [Lions 1972] or are governed by nonlinear partial differential equations either of an elliptic type as in [Bors & Walczak 2005] or of a hyperbolic type as in [Al-Hawasy 2008] or by a parabolic type as in [Chryssoverghi & Al-Hawasy 2010], or optimal control problem is governed by a couple of nonlinear partial differential equations of elliptic type [Al-Hawasy & Al-Rawdhanee 2014].

This work is concerned at first with the existence and uniqueness of the state vector solution of a couple nonlinear parabolic differential equations using the Galerkin method for a given (fixed) continuous classical control vector. Second the existence theorem of a continuous classical optimal control vector governed by the considered couple of nonlinear partial differential equation of parabolic type with equality and inequality state vector constraints is proved. The existence and uniqueness solution of the couple of adjoint vector equations associated with the considered couple equations of the state is studied. The Fréchet derivative of the Hamiltonian of this problem is derived. Finally the theorems of necessary and sufficient conditions of optimality of the problem are proved.

2. Description of the problem

Let $I=(0,T)$, $T<\infty$, $\Omega\subset R^2$ be an open and bounded region with Lipschitz boundary $\partial\Omega$, $Q=\Omega\times I$, $\Sigma=\partial\Omega\times I$. Consider the following continuous classical optimal control problem: The state equations are given by the non linear parabolic equations:

$y_{1t}-\Delta y_1+y_1-y_2=f_1(x,t,y_1,u_1)$, ... (1)

$y_{2t}-\Delta y_2+y_2+y_1=f_2(x,t,y_2,u_2)$, ... (2)

$y_1(x,t)=0$, on Σ .... (3)

$y_1(x,0)=y_1^0(x)$, on Ω .... (4)

$y_2(x,t)=0$, on Σ .... (5)

$y_2(x,0)=y_2^0(x)$, on Ω .... (6)

where $\vec{y}=(y_1,y_2)\in(C^2(\mathrm{Q}))^2$ is the state vector $\vec{u}=(u_1,u_2)\in(L^2(\mathrm{Q}))^2$ is the classical control vector and $(f_1,f_2)\in(L^2(\mathrm{Q}))^2$ is a vector of a given function defined on $\Omega\times\mathbb{R}\times U_1$ and $\Omega\times\mathbb{R}\times U_2$ respectively with $U_1\subset\mathbb{R}$

and $U_2 \subset \mathbb{R}$. The controls constraints (the control set) are $\vec{u} \in \vec{W}$, $\vec{W} \subset \left(L^2(Q)\right)^2$. Where $\vec{W} = \vec{W}_{\vec{U}}$ with $\vec{U} \subset \mathbb{R}^2$ is defined by

$\vec{W}_{\vec{U}} = \{\vec{w} \in \left(L^2(Q)\right)^2 | \vec{w} \in \vec{U}$, a.e. in $Q\}$,

The cost function is

$G_0(\vec{u}) = \int_Q g_{01}(x,t,y_1,u_1)dxdt + \int_Q g_{02}(x,t,y_2,u_2)dxdt$ ....(7a)

The equality and inequality constraints are

$G_1(\vec{u}) = \int_Q g_{11}(x,t,y_1,u_1)dxdt + \int_Q g_{12}(x,t,y_2,u_2)dxdt = 0$ .... (7b)

$G_2(\vec{u}) = \int_Q g_{21}(x,t,y_1,u_1)dxdt + \int_Q g_{22}(x,t,y_2,u_2)dxdt \leq 0$ ....(7c)

The set of admissible control is

$\vec{W}_A = \left\{\vec{u} \in \vec{W} | G_1(\vec{u}) = 0, G_2(\vec{u}) \leq 0\right\}$

The continuous optimal control problem is to minimize the cost functional (7a) subject to the constraints (7b&c), i.e. to find $\vec{u} \in \vec{W}_A$ such that $G_0(\vec{u}) = \underset{\vec{w}\in\vec{W}_A}{Min}\, G_0(\vec{w})$

Let $\vec{V} = V_1 \times V_2 = \left\{\vec{v}: \vec{v} \in \left(H^1(\Omega)\right)^2, v_1 = v_2 = 0 \text{ on } \partial\Omega\right\}$, $\vec{v} = (v_1, v_2)$. We denote by (v,v) and $\|v\|_0$ the inner product and the norm in $L^2(\Omega)$, by $(v,v)_1$ and $\|v\|_1$ the inner product and the norm in $H^1(\Omega)$, by $(\vec{v},\vec{v})$ and $\|\vec{v}\|_0$ the inner product and the norm in $L^2(\Omega) \times L^2(\Omega)$ by $(\vec{v},\vec{v})_1 = (v_1,v_1)_1 + (v_2,v_2)_1$ and $\|\vec{v}\|_1 = \|v_1\|_1 + \|v_2\|_1$ the inner product and the norm in $\vec{V}$ and $\vec{V}^*$ is the dual of $\vec{V}$.

The weak form of the problem (1-6) when $\vec{y} \in (H_0^1(\Omega))^2$ is given by

$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) = (f_1, v_1), \forall v_1 \in V_1$ ....(8a)

$(y_1^0, v_1) = (y_1(0), v_1)$, ... (8b)

$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_1, v_2) = (f_2, v_2), \forall v_2 \in V_2$ ....(9a)

$(y_2^0, v_2) = (y_2(0), v_2)$ .... (9b)

The following assumptions are necessary to study the classical optimal control problem:

Assumptions (A): $\forall i = 1,2$, assume that

i) f_i is of the Carathéodory type on $Q \times (R \times R)$, satisfies the following condition with respect to y_i& u_i,i.e. for $(x,t) \in Q$

$|f_i(x,t,y_i,u_i)| \leq \eta_i(x,t) + c_i|y_i| + \acute{c}_i|u_i|$

Where $y_i, u_i \in R$,$c_i, \acute{c}_i > 0$, $\eta_i \in L^2(Q,R)$

ii) f_i is satisfied Lipschitz condition with respect to y_i, i.e. for $(x,t) \in Q$

$|f_i(x,t,y_i,u_i) - f_i(x,t,\bar{y}_i,u_i)| \leq L_i|y_i - \bar{y}_i|$, where $y_i, \bar{y}_i, u_i \in R$ & $L_i > 0$.

3. The Solution of the State Equations

In this section the existence theorem of a unique solution of the coupled of nonlinear partial differential equations of parabolic type under a suitable assumption is proved when the control vector is given.

Proposition 3.1 [Chryssovergh, 2003]:

Suppose D be a measurable subset of $\mathbb{R}^d$ ($d = 2,3$), $f: D \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of Carathéodory type satisfies$\|f(v,x)\| \leq \xi(v) + \eta(v)\|x\|^\alpha$, for each $(v,x) \in D \times \mathbb{R}^n$, where $x \in L^p(D,\mathbb{R}^n)$, $\xi \in L^1(D,R)$, $\eta \in L^{\frac{p}{P-\alpha}}(D,R)$,$\alpha \in [0,p]$, if $p \neq \infty$, $\eta = 0$ if $p = \infty$. Then the functional $F(x) = \int_D f\left(v, x(v)\right)dv$ is continuous.

Theorem 3.1: ***(Existence and Uniqueness of Solution of the State Equations):***

With assumptions (A), for each fixed $\vec{u} \in \left(L^2(Q)\right)^2$, the weak form (8-9) of the state equations has a unique solution $\vec{y} = (y_1, y_2)$, $\vec{y}_t = (y_{1t}, y_{2t})$, $\vec{y} \in \left(L^2(I,V)\right)^2$, $\vec{y}_t \in \left(L^2(I,V^*)\right)^2$.

***Proof*:**

Let $V_n \subset V$ be the set of continuous and piecewise affine function in Ω. Let $\vec{v}_n = (v_{1n}, v_{2n})$ with $v_{in} \in V_n$, $\forall i = 1,2$ and $\vec{y}_n = (y_{1n}, y_{2n})$, $\forall n$

$y_{1n} = \sum_{j=1}^n c_{1j}(t) v_{1j}(x)$ .... (10)

& $y_{2n} = \sum_{j=1}^n c_{2j}(t) v_{2j}(x)$ .... (11)

where $c_{ij}(t)$ is unknown function of t, for each $= 1,2$, $j = 1,2,\dots,n$.

The weak forms of the state equations (8-9) are approximated w.r.t. x using the Galerkin's method, hence they become

$\langle y_{1nt}, v_1\rangle + (\nabla y_{1n}, \nabla v_1) + (y_{1n}, v_1) - (y_2, v_1) = (f_1(y_{1n}, u_1), v_1), \forall v_1 \in V_n$ .....(12a)

$(y_{1n}^0, v_1) = (y_1^0, v_1), \forall v_1 \in V_n$ .... (12b)

$\langle y_{2nt}, v_2\rangle + (\nabla y_{2n}, \nabla v_2) + (y_{2n}, v_2) + (y_1, v_2) = (f_2(y_{2n}, u_2), v_2), \forall\, v_2 \in V_n$ ...(13a)

$(y_{2n}^0, v_2) = (y_2^0, v_2), \forall\, v_2 \in V_n$ .... (13b)

where $y_{in}^0 = y_{in}^0(x) = y_{in}(x,0) \in V_n$ is the projection of $y_i^0 \in L^2(\Omega)$, i.e.,$\forall i = 1,2$

$(y_{in}^0, v_i) = (y_i^0, v_i)\ \forall\, v_i \in V_n \Leftrightarrow$
$\|y_{in}^0 - y_i^0\|_0 \le \|y_i^0 - v_i\|_0, \forall v_i \in V_n.$

Substituting (10) in (12 a& b) and (11) in (13 a & b) respectively and setting $v_1 = v_{1i}$, $v_2 = v_{2i}$, the obtained equations are equivalent to the following 1st order nonlinear system of ordinary differential equations with their initial conditions which has a unique solution y_{1n}&y_{2n} **[**Brauer, 1973**]:**

$AC_1^{'}(t) + DC_1(t) - EC_2(t) = b_1\,(\bar{V}_1^T c_1(t))$
$AC_1(0) = b_1^0$
$BC_2^{'}(t) + FC_2(t) + HC_1(t) \ = b_2\,(\bar{V}_2^T c_2(t))$
$BC_2(0) = b_2^0$
where, $C_l(t) = \left(c_{lj}(t)\right)_{n\times 1}$, $C_l^{'}(t) = \left(c_{lj}^{'}(t)\right)_{n\times 1}$, $C_l(0) = \left(c_{lj}(0)\right)_{n\times 1}$,
$b_l = (b_{li})_{n\times 1}$, $\quad b_{li} = (f_l(V_l^T c_l(t), u_l), v_{li})$,
$b_l^0 = (b_{lj}^0)$, $\ b_{lj}^0 = (y_l^0, v_{lj})$, $\ \forall l = 1,2$, $\ A = \left(a_{ij}\right)_{n\times n}$, $\ a_{ij} = \left(v_{1j}, v_{1i}\right)$, $\ E = \left(e_{ij}\right)_{n\times n}$,
$e_{ij} = \left(v_{2j}, v_{1i}\right)$, $B = \left(b_{ij}\right)_{n\times n}$,
$b_{ij} = \left(v_{2j}, v_{2i}\right)$, $D = \left(d_{ij}\right)_{n\times n}$,
$d_{ij} = [\left(\nabla v_{1j}, \nabla v_{1i}\right) + \left(v_{1j}, v_{1i}\right)]$,
$F = \left(f_{ij}\right)_{n\times n}$, $f_{ij} = [\left(\nabla v_{2j}, \nabla v_{2i}\right) + \left(v_{2j}, v_{2i}\right)]$, and $H = \left(h_{ij}\right)_{n\times n}$,
$h_{ij} = (v_{1i}, v_{2i})$.

Now, to show the norm $\|\vec{y}_n^0\|_0$ is bounded: Since $\vec{y}^0 \in (L^2(\Omega))^2$, then there exists $\{\vec{v}_n^0\}$, with $\vec{v}_n^0 \in \vec{V}_n$ such that $\vec{v}_n^0 \longrightarrow \vec{y}^0$ strongly in $(L^2(\Omega))^2$ then from the projection theorem and (12b&13b) one obtain that $\vec{y}_n^0 \longrightarrow \vec{y}^0$ strongly in $(L^2(\Omega))^2$with $\|\vec{y}_n^0\|_{\mathbf{0}} \le\ b_1$

The norms $\|\vec{y}_n(t)\|_{L^\infty(I,L^2(\Omega))}$ and $\|\vec{y}_n(t)\|_Q$ are bounded: Setting $v_1 = y_{1n}$ and $v_2 = y_{2n}$ in (12a) & (13a) respectively, integrating from 0 to T, adding the two obtained equations one get

$\int_0^T \langle \vec{y}_{nt}, \vec{y}_{1n}\rangle dt + \int_0^T \|\vec{y}_n\|_1^2 dt =$
$\int_0^T [(f_1(y_{1n}, u_1), y_{1n}) + (f_2(y_{2n}, u_2), y_{2n})]dt$ .... (14)

Since the 2nd term of L.H.S. of (14) is positive, then using Lemma 1.2 in **[**Temam, 1977**]** for the 1st term of it, taking $T = t \in [0, T]$, finally applying assumptions(A-i) for the R.H.S. of (14), i.e.

$\int_0^t \frac{d}{dt}\|\vec{y}_n(t)\|_0^2\, dt$
$\le \int_0^t \int_\Omega (\eta_1^2 + |y_{1n}|^2)\, dxdt +$
$2\int_0^t \int_\Omega c_1 |y_{1n}|^2\, dxdt +$
$\int_0^t \int_\Omega (\acute{c}_1 |u_1|^2 + |y_{1n}|^2) dxdt +$
$\int_0^t \int_\Omega (\eta_2^2 + |y_{2n}|^2)\, dxdt +$
$2\int_0^t \int_\Omega c_2 |y_{2n}|^2\, dxdt +$
$\int_0^t \int_\Omega (\acute{c}_2 |u_2|^2 + |y_{2n}|^2) dxdt,$ .... (15)

Since $\|\eta_i\|_Q \le \acute{b}_i, \|u_i\|_Q \le c_{i1},\ \forall i = 1,2$ and $\|\vec{y}_n(0)\|_0^2 \le b$, then (15) becomes $\|\vec{y}_n(t)\|_0^2 \le c^* + c_5 \int_0^t \|\vec{y}_n\|_0^2\, dt$,

where $c^* = b + b_1^{'} + b_2^{'} + \acute{c}_1 c_{11} + \acute{c}_1 \acute{c}_{12}$,
$c_5 = 2 + c_3 + c_4$, with $c_3 = 2c_1$, $c_4 = 2c_2$.
Using Belman- Gronwall inequality, to get $\|\vec{y}_n(t)\|_{\mathbf{0}}^2 \le c^* e^{c_5} = b^2(c), \forall\, t \in [0, T]$, easily the following are obtained
$\|\vec{y}_n(t)\|_{L^\infty(I,L^2(\Omega))} \le b(c)$, and $\|\vec{y}_n(t)\|_{\boldsymbol{Q}} \le b_1(c)$.

The norm $\|\vec{y}_n(t)\|_{L^2(I,V)}$ is bounded:
Again using the same above steps in (14), but with $t = T$, and $\|\vec{y}_n(T)\|_{\mathbf{0}}^2$ is positive, one easily obtain that
$\|\vec{y}_n\|_{L^2(I,V)} = \int_0^T \|\vec{y}_n\|_1^2\, dt \le\ b_2^2(c)$,
where $b_2^2(c) = \frac{\left(b + b_1^{'} + b_2^{'} + \acute{c}_1 c_1 + \acute{c}_2 c_2 + c_5 b_1(c)\right)}{2}$

The convergence of the solution:
Let $\{\vec{V}_n\}_{n=1}^\infty$ be a sequence of subspaces of $\vec{V}$, such that $\forall\, \vec{v} = (v_1, v_2) \in\ \vec{V}$, there exists a sequence $\{\vec{v}_n\}$ with $\vec{v}_n = (v_{1n}, v_{2n}) \in \vec{V}_n, \forall n$, and $\vec{v}_n \longrightarrow \vec{v}$ strongly in $\vec{V} \Rightarrow \vec{v}_n \longrightarrow \vec{v}$ strongly in $\left(L^2(\Omega)\right)^2$. Then corresponding to the sequence $\{\vec{V}_n\}_{n=1}^\infty$, one obtain a sequence of approximation problems like (12 a & b) and (13 a &b), but with $\vec{v} = \vec{v}_n = (v_{1n}, v_{2n})$ for $n = 1,2, \dots$, and $y_{1n}, y_{2n} \in L^2(I, V_n)$ a.e. in I, i.e

$\langle y_{1nt}, v_{1n}\rangle + (\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) = (f_1(y_{1n}, u_1), v_{1n})$, ...(16a)
$(y_{1n}^0, v_{1n}) = (y_1^0, v_{1n})$, ... (16b)
& $\langle y_{2nt}, v_{2n}\rangle + (\nabla y_{2n}, \nabla v_{2n}) + (y_{2n}, v_{2n}) + (y_{1n}, v_{2n}) = (f_2(y_{2n}, u_2), v_{2n})$, ...(17a)

$(y_{2n}^0, v_{2n}) = (y_2^0, v_{2n})$, ... (17b)

which has a sequence of solutions $\{\vec{y}_n\}_{n=1}^{\infty}$, where $\vec{y}_n = (y_{1n}, y_{2n})$.
Since $\|\vec{y}_n\|_{L^2(Q)}$ and $\|\vec{y}_n\|_{L^2(I,V)}$ are bounded, then by Alaoglu's theorem, there exists a subsequence of $\{\vec{y}_n\}_{n\in N}$, say again $\{\vec{y}_n\}_{n\in N}$ such that $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\left(L^2(Q)\right)^2$ and weakly in $\left(L^2(I,V)\right)^2$. In this point, it is required to show that the norm $\|\vec{y}_{kt}\|_{L^2(I,V^*)}$ is bounded, but this will be left here and will be shown in section 4, so assume it is bounded, and since

$$\left(L^2(R,V)\right)^2 \subset \left(L^2(R,\Omega)\right)^2 \cong \left(\left(L^2(R,\Omega)\right)^*\right)^2 \subset \left(L^2(R,V^*)\right)^2 \quad \text{................ (18)}$$

Which means the injection of $\left(L^2(R,V)\right)^2$ in to $\left(L^2(R,\Omega)\right)^2$, and of $\left(\left(L^2(R,\Omega)\right)^*\right)^2$ in to $\left(L^2(R,V^*)\right)^2$ are continuous, the injection of $\left(L^2(R,V)\right)^2$ in to $\left(L^2(Q)\right)^2$ is compact, on the other hand from assumptions (A), the Cauchy-Schwartz inequality, Fourier transform and its inverse and finally the Parseval theorem, the compactness theorem [Temam, 1977] can be applied to get that there exists a subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ such that $\vec{y}_k \longrightarrow \vec{y}$ strongly in $\left(L^2(Q)\right)^2$.
Now, multiplying both sides of (16a) and (17a) by $\varphi_i(t) \in C^1[0,T]$, such that $\varphi_i(T) = 0$, $\forall i = 1,2$, taking the integra from 0 to T, finally using integration by parts for the 1st term of each one of the obtained equation, yield

$$-\int_0^T (y_{1n}, v_{1n})\varphi_1'(t)dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n})\varphi_1(t) + (y_{1n}, v_{1n})\varphi_1(t) - (y_{2n}, v_{1n})\varphi_1(t)]dt = \int_0^T (f_1(y_{1n}, u_1), v_{1n})\varphi_1(t)dt + (y_{1n}^0, v_{1n})\varphi_1(0), \quad \text{..................................... (19)}$$

$$\& -\int_0^T (y_{2n}, v_{2n})\varphi_2'(t)dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n})\varphi_2(t) + (y_{2n}, v_{2n})\varphi_2(t) + (y_{1n}, v_{2n})\varphi_2(t)]dt = \int_0^T (f_2(y_{2n}, u_2), v_{2n})\varphi_2(t)dt + (y_{2n}^0, v_{2n})\varphi_2(0) \quad \text{.................................... (20)}$$

Since $\forall i = 1,2$ the following converges hold

$$\left.\begin{array}{l} v_{in} \longrightarrow v_i \text{ strongly in } V \\ v_{in} \longrightarrow v_i \text{ strongly in } L^2(\Omega) \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{l} v_{in}\varphi_i \longrightarrow v_i\varphi_i \text{ strongly in } L^2(I,V) \\ v_{in}\varphi_i' \longrightarrow v_i\varphi_i' \text{ strongly in } L^2(Q) \end{array}\right.$$

$y_{in} \longrightarrow y_i$ weakly in $L^2(Q)$ & in $L^2(I,V)$ and $y_{in}^0 \longrightarrow y_i^0$, strongly in $L^2(\Omega)$, then

$$\int_0^T (y_{1n}, v_{1n})\varphi_1'(t)dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n})\varphi_1(t) + (y_{1n}, v_{1n})\varphi_1(t) - (y_{2n}, v_{1n})\varphi_1(t)]dt \longrightarrow \int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t) + (y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t)]dt \quad \text{..........(21a)}$$

$$(y_{1n}^0, v_{1n})\varphi_1(0) \to (y_1^0, v_1)\varphi_1(0) \quad \text{.......... (21b)}$$

$$\& \int_0^T (y_{2n}, v_{2n})\varphi_2'(t)dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n})\varphi_2(t) + (y_{2n}, v_{2n})\varphi_2(t) + (y_{1n}, v_{2n})\varphi_2(t)]dt \longrightarrow \int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)]dt \quad \text{...........(22a)}$$

$$(y_{2n}^0, v_{2n})\varphi_2(0) \longrightarrow (y_2^0, v_2)\varphi_2(0) \quad \text{........ (22b)}$$

On the other hand, let $w_{in} = v_{in}\varphi_i$ and $w_i = v_i\varphi_i$ then $\forall i = 1,2$, $w_{in} \longrightarrow w_i$ strongly in $L^2(Q)$ and then w_{in} is measurable w.r.t. (x,t), using assumption (A-i), then applying Proposition 1.3, the integral $\int_Q f_i(x,t,y_{in},u_i)w_{in}dxdt$ is continuous w.r.t. (y_{in}, u_i, w_{in}), but $y_{in} \longrightarrow y_i$ strongly in $L^2(Q)$, then $\forall i = 1,2$

$$\int_0^T (f_i(y_{in}, u_i), v_{in})\varphi_i(t)dt \to \int_0^T (f_i(y_i, u_i), v_i)\varphi_i(t)dt \quad \text{(21c)}$$

From (21a,b &c) and (22a&b) then (19) and (20) become respectively

$$-\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t) + (y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t)]dt = \int_0^T (f_1(y_1, u_1), v_1)\varphi_1(t)dt + (y_1^0, v_1)\varphi_1(0) \quad \text{................................ (23)}$$

$$\& \int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)]dt = \int_0^T (f_2(y_2, u_2), v_2)\varphi_2(t)dt + (y_2^0, v_2)\varphi_2(0) \quad \text{................................ (24)}$$

Case1:

Choose $\varphi_i \in D[0,T]$, i.e., $\varphi_i(0) = \varphi_i(T) = 0$, $\forall i = 1,2$, substituting in (23) and (24), and integration by parts for the 1^{st} terms in the L.H.S. of each one of the obtained equation, yield

$\int_0^T \langle y_{1t}, v_1\rangle \varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t)$
$+(y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t)]dt$
$= \int_0^T (f_1(y_1, u_1), v_1)\varphi_1(t)dt$ (25)

$$\int_0^T \langle y_{2t}, v_2\rangle \varphi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)]dt$$
$= \int_0^T (f_2(y_2, u_2), v_2)\varphi_2(t)dt$ .... (26)

which give that y_1 & y_2 are solutions of (8a) and (9a) respectively (a.e. on I)

Case2:

Choose $\varphi_i \in C^1[0,T]$, such that $\varphi_i(T) = 0$ & $\varphi_i(0) \neq 0, \forall i = 1,2$. Using integration by parts for 1^{st} term in the L.H.S. of (25) & (26), subtracting (23) & (24) from the equations which are obtained from (25) & (26) respectively, one get

$(y_i^0, v_i)\varphi_1(0) = (y_i(0), v_i)\varphi_1(0),$ .... (27)

which give the i. cs. (8b)& (9b) are hold.

The strong convergence for $\vec{y}_n$:

Let $a_i(y_{in}, y_{in}) = (\nabla y_{in}, \nabla y_{in}) + (y_{in}, y_{in})$ .... (28)

and

$a(\vec{y}_n, \vec{y}_n) = a_1(y_{1n}, y_{1n}) + a_2(y_{2n}, y_{2n})$ .. (29)

For each $i = 1,2$ Substituting $v_1 = y_{1n}$ and $v_2 = y_{2n}$ in (12a) and (13a) respectively, integrating both sides of the above two obtained equations from 0 to T, then adding both of them, one has

$$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n\rangle\, dt + \int_0^T a(\vec{y}_n, \vec{y}_n)dt =$$
$\int_0^T [(f_1(y_{1n}, u_1), y_{1n})\, dt$
$+ \int_0^T (f_2(y_{2n}, u_2), y_{2n})\, dt$ ....(30a)

Also, the same above steps are repeated but with substituting $v_1 = y_1$ and $v_2 = y_2$ in (8a) and (9a) respectively, to get

$\int_0^T \langle \vec{y}_t, \vec{y}\rangle\, dt + \int_0^T a(\vec{y}, \vec{y})dt =$
$\int_0^T [(f_1(y_1, u_1), y_1) + (f_2(y_2, u_2), y_2)]dt$ .... (30b)

Again, using Lemma 1.2 in [Temam, 1977], the 1^{st} terms in the L.H.S. of (30a&b), yield

$\frac{1}{2}\|\vec{y}_n(T)\|_0^2 - \frac{1}{2}\|\vec{y}_n(0)\|_0^2 + \int_0^T a(\vec{y}_n, \vec{y}_n)dt$
$= \int_0^T (f_1(y_{1n}, u_1), y_{1n})dt$
$+ \int_0^T (f_2(y_{2n}, u_2), y_{2n})dt$ ....(31a)

$$\frac{1}{2}\|\vec{y}(T)\|_0^2 - \frac{1}{2}\|\vec{y}(0)\|_0^2 + \int_0^T a(\vec{y}, \vec{y})dt =$$
$\int_0^T [(f_1(y_1, u_1), y_1) + (f_2(y_2, u_2), y_2)]\, dt$ .... (31b)

Since

$\frac{1}{2}\|\vec{y}_n(T) - \vec{y}(T)\|_0^2 - \frac{1}{2}\|\vec{y}_n(0) - \vec{y}(0)\|_0^2 +$
$\int_0^T a(\vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})\, dt$
$=$ (32a) $-$ (32b) $-$ (32c) (32)

where

(32a)= $\frac{1}{2}\|\vec{y}_n(T)\|_0^2 - \frac{1}{2}\|\vec{y}_n(0)\|_0^2$
$+ \int_0^T a(\vec{y}_n(t), \vec{y}_n(t))dt$

(32b)= $\frac{1}{2}(\vec{y}_n(T), \vec{y}(T)) - \frac{1}{2}(\vec{y}_n(0), \vec{y}(0)) +$
$\int_0^T a(\vec{y}_n(t), \vec{y}(t))\, dt$, and

(32c)= $\frac{1}{2}(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)) -$
$\frac{1}{2}(\vec{y}(0), \vec{y}_n(0) - y(0)) +$
$\int_0^T a(\vec{y}(t), \vec{y}_n(t) - \vec{y}(t))$

Since $\vec{y}_n^0 = \vec{y}_n(0)$, & $\vec{y}^0 = \vec{y}(0)$, then

$$\left.\begin{array}{l}\vec{y}_n(0) \longrightarrow \vec{y}(0) \text{ strongly in} \left(L^2(\Omega)\right)^2 \\ \vec{y}_n(T) \longrightarrow \vec{y}(T) \text{strongly in} \left(L^2(\Omega)\right)^2\end{array}\right\} \text{...(33a)}$$

$$\left.\begin{array}{l}(\vec{y}(0), \vec{y}_n(0) - \vec{y}(0)) \to 0 \\ (\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)) \to 0\end{array}\right\} \text{.................. (33b)}$$

and $$\left.\begin{array}{l}\|\vec{y}_n(0) - \vec{y}(0)\|_0^2 \to 0 \\ \|\vec{y}_n(T) - \vec{y}(T)\|_0^2 \to 0\end{array}\right\} \text{....................(33c)}$$

But $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\left(L^2(I, V)\right)^2$, then

$\int_0^T \vec{a}(\vec{y}(t), \vec{y}_n(t) - \vec{y}(t))dt \to 0$ .... (33d)

Since the integral $\int_0^T (f_i(y_{in}, u_i), y_{in})\, dt$ is continuous w.r.t. y_i & u_i $\forall i = 1,2$ and $\vec{y}_n \longrightarrow \vec{y}$, strongly in $\left(L^2(Q)\right)^2$, one get that

$\int_0^T [(f_1(y_{1n}, u_1), y_{1n}) +$
$(f_2(y_{2n}, u_2), y_{2n})]\, dt \to \int_0^T [(f_1(y_1, u_1), y_1) +$
$(f_2(y_2, u_2), y_2)]dt$ ....(33e)

Now, when $n \to \infty$ in both sides of (32), one have the following results:

(1)The first two terms in the L.H.S. of (32) are tending to zero (from 33c)

(2) from (31a), Eq.(32a)=

$\int_0^T [(f_1(y_{1n},u_1),y_{1n}) + (f_2(y_{2n},u_2),y_{2n})]\,dt$
$\to \int_0^T [(f_1(y_1,u_1),y_1) + (f_2(y_2,u_2),y_2)]\,dt$

(3) Eq.(32b) $\to$ L.H.S. of (31b) $= \int_0^T [(f_1(y_1,u_1),y_1) + (f_2(y_2,u_2),y_2)]\,dt$

(4) The 1^{st} two terms in (32c) are tending to zero (from (33b)), and the last one term also is tended to zero (from (33d)).

Now, substituting these results in (32) with n tends to ∞, gives

$$\int_0^T \|\vec{y}_n - \vec{y}\|_1^2\,dt = \int_0^T a(\vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})dt \to 0$$

$\Rightarrow\ \vec{y}_n \longrightarrow \vec{y}$ strongly in $\left(L^2(I,V)\right)^2$.

Uniqueness of the Solution:

Let $\vec{y} = (y_1,y_2)$ and $\vec{\bar{y}} = (\bar{y}_1,\bar{y}_2)$ are two solutions of the weak form (8a-9a), i.e. y_1 and $\bar{y}_1$ are satisfied the weak form (8a), subtracting each equation from the other and then setting $v_1 = y_1 - \bar{y}_1$, yields to

$$\langle (y_1-\bar{y}_1)_t, y_1-\bar{y}_1\rangle + \|y_1-\bar{y}_1\|_1^2 = (f_1(y_1,u_1) - f_1(\bar{y}_1,u_1), y_1-\bar{y}_1)$$

The same thing, for y_2 and $\bar{y}_2$, one have that

$$\langle (y_2-\bar{y}_2)_t, y_2-\bar{y}_2\rangle + \|y_2-\bar{y}_2\|_1^2 = (f_2(y_2,u_2) - f_2(\bar{y}_2,u_2), y_2-\bar{y}_2)$$

Adding the above two equations, since the 2^{nd} term of the L.H.S. of the obtained equation is positive, using Lemma 1.2 in ref. [Temam, 1977] for the remained 1^{st}in L.H.S., integrating both sides from 0 to t, applying the Lipshctiz property on the R.H.S., and finally the Bellamn-Gronwal inequality, gives $\left\|(\vec{y}-\vec{\bar{y}})(t)\right\|_0^2 = 0, \forall t \in I$.

Now, repeating the above steps but with keeping the 2^{nd}term of the L.H.S. and integrating from 0 to T, using the initial condition, one have $\int_0^T \left\|\vec{y}-\vec{\bar{y}}\right\|_1^2 dt \le L\int_0^T \left\|\vec{y}-\vec{\bar{y}}\right\|_0^2 dt \le 0$

$\Rightarrow\ \left\|\vec{y}-\vec{\bar{y}}\right\|_{L^2(I,V)} = 0 \Rightarrow \vec{y} = \vec{\bar{y}}$

Lemma 3.1:

(a) In addition to assumptions (A), if $\vec{y}$ and $\vec{y} + \overrightarrow{\delta y}$ are the states vectors corresponding to the controls vectors $\vec{u} \in (L^2(Q))^2$ and $\vec{u} + \overrightarrow{\delta u} \in (L^2(Q))^2$, then $\left\|\overrightarrow{\delta y}\right\|_{L^\infty\left(I,L^2(\Omega)\right)} \le M\left\|\overrightarrow{\delta u}\right\|_Q$,

$\left\|\overrightarrow{\delta y}\right\|_{L^2(Q)} \le M\left\|\overrightarrow{\delta u}\right\|_Q$ & $\left\|\overrightarrow{\delta y}\right\|_{L^2(I,V)} \le M\left\|\overrightarrow{\delta u}\right\|_Q$

(b) In addition to assumptions (A), the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ from $(L^2(Q))^2$ into $(L^\infty(I,L^2(\Omega)))^2$ or in to $(L^2(I,V))^2$ or in to $(L^2(Q))^2$ is continuous.

***Proof*:**

(a) Let$\vec{u} = (u_1,u_2)$,$\vec{\bar{u}} = (\bar{u}_1,\bar{u}_2) \in (L^2(Q))^2$ then by theorem 3.1, there exist their corresponding states solutions $\vec{y} = (y_1,y_2)$, $\vec{\bar{y}} = (\bar{y}_1,\bar{y}_2)$, which are satisfied the weak forms (8 a& b) and (9 a& b) respectively, setting $\delta y_1 = \bar{y}_1 - y_1$, $\delta y_2 = \bar{y}_2 - y_2$, $\delta u_1 = \bar{u}_1 - u_1$ & $\delta u_2 = \bar{u}_2 - u_2$, once get

$\langle \delta y_{1t}, v_1\rangle + (\nabla\delta y_1, \nabla v_1) + (\delta y_1, v_1) - (\delta y_2, v_1) = (f_1(y_1+\delta y_1, u_1+\delta u_1), v_1)$
$-(f_1(y_1,u_1),v_1)$ (34a)

& $\langle \delta y_{2t}, v_2\rangle + (\nabla\delta y_2, \nabla v_2) + (\delta y_2, v_2) + (\delta y_1, v_2) = (f_2(y_2+\delta y_2, u_2+\delta u_2), v_2)$
$-(f_2(y_2,u_2),v_2)$ (34b)

Substituting $v_1 = \delta y_1$ in (34a) and $v_2 = \delta y_2$ in (34b), adding the two equations, yields

$$\frac{1}{2}\frac{d}{dt}\left\|\overrightarrow{\delta y}\right\|_0^2 + \left\|\overrightarrow{\delta y}\right\|_1^2 = (f_1(y_1+\delta y_1, u_1+\delta u_1) - f_1(y_1,u_1), \delta y_1 + (f_2(y_2+\delta y_2, u_2+\delta u_2) - f_2(y_2,u_2), \delta y_2) \qquad \text{.................................(34c)}$$

The 2^{nd} term of L.H.S. is positive, using Lemma 1.2 in the. ref. [Temam, 1977] for the remainder term, integrating from 0 to t, using the Lipshctiz property for the terms in the R.H.S., one get,$\forall t \in [0,T]$

$$\left\|\overrightarrow{\delta y}(t)\right\|_0^2 \le 2\int_0^t\int_\Omega [L_1|\delta y_1|^2 + \bar{L}_1|\delta u_1||\delta y_1|]dxdt + 2\int_0^t\int_\Omega [L_2|\delta y_2|^2 + \bar{L}_2|\delta u_2||\delta y_2|]dxdt +$$

$$\le 2L_1\int_0^t \|\delta y_1\|_0^2\,dt + \bar{L}_1\int_0^T \|\delta u_1\|_0^2\,dt + \bar{L}_1\int_0^t \|\delta y_1\|_0^2\,dt + 2L_2\int_0^t \|\delta y_2\|_0^2\,dt + \bar{L}_2\int_0^T \|\delta u_2\|_0^2\,dt + \bar{L}_2\int_0^t \|\delta y_2\|_0^2\,dt$$

$$\Rightarrow \left\|\overrightarrow{\delta y}(t)\right\|_0^2 \le 2\tilde{L}_1\left\|\overrightarrow{\delta u}\right\|_Q^2 + \tilde{L}_2\int_0^t \left\|\overrightarrow{\delta y}\right\|_0^2 dt$$

Where $\tilde{L}_1 = \bar{L}_1 + \bar{L}_2$, $\tilde{L}_2 = 2(L_1 + L_2) + \tilde{L}_1$
The Belman-Gronwall inequality, gives $\left\|\overrightarrow{\delta y}(t)\right\|_0^2 \leq M^2 \left\|\overrightarrow{\delta u}\right\|_Q^2$, where $M^2 = \tilde{L}_1 e^{\tilde{L}_2 T}$

$\Rightarrow \left\|\overrightarrow{\delta y}(t)\right\|_0 \leq M \left\|\overrightarrow{\delta u}\right\|_Q, t \in [0,T]$

$\Rightarrow \left\|\overrightarrow{\delta y}\right\|_{L^\infty\left(I, L^2(\Omega)\right)} \leq M \left\|\overrightarrow{\delta u}\right\|_Q$, which gives

$\left\|\overrightarrow{\delta y}\right\|_{L^2(Q)} \leq M \left\|\overrightarrow{\delta u}\right\|_Q, M^2 = \bar{M}^2 = TM^2$

Using the same above steps in (34c) but with $t = T$, i.e.

$\left\|\overrightarrow{\delta y}(T)\right\|_0^2 + 2\int_0^T \left\|\overrightarrow{\delta y}\right\|_1^2 dt$
$\leq \tilde{L}_1 \left\|\overrightarrow{\delta u}\right\|_Q^2 + \tilde{L}_2 \left\|\overrightarrow{\delta y}\right\|_Q^2 \Rightarrow$
$\left\|\overrightarrow{\delta y}\right\|_{L^2(I,V)} \leq M \left\|\overrightarrow{\delta u}\right\|_Q$,
where $M^2 = \bar{M}^2 = (\tilde{L}_1 + \tilde{L}_2 M^2)/2$

(b) The Lipschitz continuous of $\vec{u} \longmapsto \vec{y}$ easily obtained using the results in (a).

4. Existence of a Classical Optimal Control

In this section the existence theorem of a continuous classical optimal control vector satisfying the equality and inequality state constraints is studied. Therefor the following assumption and lemma are needed.

Assumptions (B): Consider g_{li} (for $l = 0,1,2$ and i$= 1,2$) is of Carathéodory type on $Q \times (R \times R)$, and satisfies the following condition w.r.t. $y_i \in R$ & $u_i \in R$

$|g_{li}(x,t,y_i,u_i)| \leq \eta_{li}(x,t) + c_{li1}(y_i)^2 + c_{li2}(u_i)^2, \eta_{li} \in L^1(Q)$.

Lemma 4.1:

With assumptions (B) the functional $\vec{u} \longmapsto G_l(\vec{u})$, for each $l = 0,1,2$; defined on $L^2(Q)$ is continuous.

***Proof*:**

Using assumption (B) and Proposition 3.1, the integral $\int_Q g_{li}(x,t,y_i,u_i)\, dxdt$ is continuous on $L^2(Q)$, $\forall\, i = 1,2$, $\forall l = 0,1,2$ hence $G_l(\vec{u})$
is continuous on $L^2(Q)$, $\forall l = 0,1,2$.

Lemma 4.2 : [Chryssoverghi 2003]

Let $g : Q \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is of Carathéodory type

on $Q \times (\mathbb{R} \times \mathbb{R})$ and satisfies:

$\left| g(x,t,y,u) \right| \leq \eta(x) + cy^2 + c'u^2$,
where $\eta(x,t) \in L^1(Q, \mathbb{R})$, $c \geq 0$ and $c' \geq 0$.

Then $\int_Q g(x,t,y,u)dx$ is continuous on $L^2(Q, \mathbb{R}^2)$, with $u \in U$, $U \subset \mathbb{R}$ is compact.

Theorem 4.1:

If $\vec{U}$ in the set of controls $\overrightarrow{W}_{\vec{U}} = W_1 \times W_2$ is convex and compact, $\overrightarrow{W}_A \neq \emptyset$, the functions f_1, f_2 with $\eta_i \in L^2(Q)$, $\forall i = 1,2$, have the form

$f_1(x,t,y_1,u_1) = f_{11}(x,t,y_1) + f_{12}(x,t)u_1$
$f_2(x,t,y_2,u_2) = f_{21}(x,t,y_2) + f_{22}(x,t)u_2$

Where $|f_{i1}(x,t,y_i)| \leq \eta_i(x,t) + c_i|y_i|$ & $|f_{i2}(x,t)| \leq k_i$, With $k_i, c_i \geq 0$, $\forall i = 1,2$. g_{1i} is independent of u_i, g_{0i} and g_{2i} are convex with respect to u_i for fixed (x,t,y_i). Then there exists a classical optimal control.

***Proof*:**

From the assumptions on $U_i \subset \mathbb{R}\ \forall i = 1,2$ and the Egorov's theorem, once get that $W_1 \times W_2$ is weakly compact. Since $\overrightarrow{W}_A \neq \emptyset$, then there exists $\vec{\vec{u}} \in \overrightarrow{W}_A$ such that $G_1(\vec{\vec{u}}) = 0, G_2(\vec{\vec{u}}) \leq 0$ and there exists a minimum sequence $\{\vec{u}_k\}$ with $\vec{u}_k \in \overrightarrow{W}_A, \forall k$, such that $\lim_{n\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\vec{u}} \in \overrightarrow{W}_A} G_0(\vec{\vec{u}})$. Since $\vec{u}_k \in \overrightarrow{W}_A, \forall k$ but $\overrightarrow{W}$ is weakly compact, there exists a subsequence of $\{\vec{u}_k\}$ say again $\{\vec{u}_k\}$ which converges weakly to some point $\vec{u}$ in $\overrightarrow{W}$, i.e. $\vec{u}_k \longrightarrow \vec{u}$ weakly in $\left(L^2(Q)\right)^2$ and $\|\vec{u}_k\|_Q \leq c, \forall k$. From theorem 3.1 the state equations has a unique solution $\vec{y}_k = \vec{y}_{\vec{u}_k}$ for each control $\vec{u}_k$, and the norms $\|\vec{y}_k\|_{L^\infty(I,L^2(\Omega))}$, $\|\vec{y}_k\|_{L^2(Q)}$ and $\|\vec{y}_k\|_{L^2(I,V)}$ are bounded, then by Alaoglu's theorem there exists a subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ which converges weakly to some point $\vec{y}$ w.r.t the above norm, i.e.

$\vec{y}_k \longrightarrow \vec{y}$ weakly in $\left(L^\infty\left(I, L^2(\Omega)\right)\right)^2$, in $\left(L^2(Q)\right)^2$ and in $\left(L^2(I,V)\right)^2$.

To show that the norm $\|\vec{y}_{kt}\|_{L^2(I,V^*)}$ is bounded, the weak forms (12a) & (13a) can rewritten in the forms

$\langle y_{1kt}, v_1 \rangle = -(\nabla y_{1k}, \nabla v_1) - (y_{1k}, v_1) + (y_{2k}, v_1) + (f_1(y_{1k}, u_{1k}), v_1)$
& $\langle y_{2kt}, v_2 \rangle = -(\nabla y_{2k}, \nabla v_2) - (y_{2k}, v_2) - (y_{1k}, v_2) + (f_2(y_{2k}, u_{2k}), v_2)$

Adding the above two equations, then integrating both sides from 0 to T, taking the absolute value and finally using the Cauchy Schwarz inequality, yields

$$\left|\int_0^T \langle \vec{y}_{kt}, \vec{v}\rangle dt\right|$$
$$\leq \|\nabla y_{1k}\|_Q \|\nabla v_1\|_Q + \|y_{1k}\|_Q \|v_1\|_Q + \|y_{2k}\|_Q \|v_1\|_Q \quad +(\|\eta_1\|_Q + c_1\|y_{1k}\|_Q + \acute{c}_1\|u_{1k}\|_Q)\|v_1\|_Q + \|\nabla y_{2k}\|_Q \|\nabla v_2\|_Q + \|y_{2k}\|_Q\|v_2\|_Q + \|y_{1k}\|_Q\|v_2\|_Q + (\|\eta_2\|_Q + c_2\|y_{2k}\|_Q + \acute{c}_2\|u_{2k}\|_Q)\|v_2\|_Q$$
$$\leq 2\|\nabla \vec{y}_k\|_Q \|\nabla \vec{v}\|_Q + 4\|\vec{y}_k\|_Q\|\vec{v}\|_Q + \bar{b}\,(c)\|\vec{v}\|_Q$$

where $\bar{b}\,(c) = \bar{b}_3(c) + \bar{b}_4(c)$, $\bar{b}_3(c) = \acute{b}_1 + c_1 b_1(c) + \acute{c}_1\bar{c}_1$ and $\bar{b}_4(c) = \acute{b}_2 + c_2 b_2(c) + \acute{c}_2\bar{c}_2$, with $\|\eta_i\|_Q \leq \acute{b}_i$, $\|y_{ik}\|_Q \leq b_i(c)$ & $\|u_{ik}\|_Q \leq \bar{c}_i$, $\forall i = 1,2$.

Setting $\tilde{b}\,(c) = 6b_2(c) + \bar{b}\,(c)$, then the above inequality $\forall \vec{y}_{kt} \in V^* \times V^*$becomes

$$\|\vec{y}_{kt}\|_{L^2(I,V^*)} = \frac{\left|\int_0^T \langle \vec{y}_{kt}, \vec{v}\rangle dt\right|}{\|\vec{v}\|_{L^2(I,V)}} \leq \tilde{b}(c),$$

Relation (18) is also satisfied here and gives that the injections of $(L^2(I,V))^2$ in to $(L^2(Q))^2$ and of $(\left(L^2(Q)\right)^*)^2$ in to $(L^2(I,V^*))^2$ are continuous and since the injections of $(L^2(I,V))^2$ in to $(L^2(Q))^2$. So we got all the hypotheses of compactness theorem, which is used here to get that there exists a subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ such that $\vec{y}_k \longrightarrow \vec{y}$ strongly in $\left(L^2(Q)\right)^2$.

Now, Since for each k, y_{1k} and y_{2k} are solutions of the weak form (12a) and (13a) respectively, substituting these solutions in the above indicate equations, then multiplying both sides of each equation by $\varphi_1(t)$ and $\varphi_2(t)$ respectively (with $\varphi_i \in C^1[0,T]$, such that $\varphi_i(T) = 0, \forall i = 1{,}2$), rewriting the 1^{st} terms in the L.H.S. of each one of their, integrating both sides from 0 to T, finally integrating by parts for these 1^{st} terms, one has

$$-\int_0^T (y_{1k}, v_1)\varphi_1'(t)dt + \int_0^T [\,(\nabla y_{1k}, \nabla v_1)\varphi_1(t) + (y_{1k}, v_1)\varphi_1(t) - (y_{2k}, v_1)\varphi_1(t)]dt$$
$$= \int_0^T (f_{11}(x,t,y_{1k}), v_1)\varphi_1(t)dt + \int_0^T (\, f_{12}(x,t)u_{1k}, v_1\varphi_1(t))\, dt + (y_{1k}(0), v_1)\varphi_1(0) \quad \ldots\ldots (35)$$

$$\& -\int_0^T (y_{2k}, v_2)\varphi_2'(t)dt + \int_0^T [\,(\nabla y_{2k}, \nabla v_2)\varphi_2(t) + (y_{2k}, v_2)\varphi_2(t) + (y_{1k}, v_2)\varphi_2(t)]dt$$
$$= \int_0^T (f_{21}(x,t,y_{2k}), v_2)\varphi_1(t)dt + \int_0^T (\, f_{22}(x,t)u_{2k}, v_2\varphi_2(t)) \; (y_{2k}(0), v_2)\varphi_2(0) \quad \ldots\ldots (36)$$

Since $\vec{y}_k \longrightarrow \vec{y}$ weakly in $\left(L^2(Q)\right)^2$ and $\vec{y}_k \longrightarrow \vec{y}$ weakly in $\left(L^2(I,V)\right)^2$, then

$$-\int_0^T (y_{1k}, v_1)\varphi_1'(t)dt + \int_0^T [\,(\nabla y_{1k}, \nabla v_1)\varphi_1(t) + (y_{1k}, v_1)\varphi_1(t) - (y_{2k}, v_1)\varphi_1(t)]dt \to \; -\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [\,(\nabla y_1, \nabla v_1)\varphi_1(t) + (y_1, v_1)\varphi_1(t) - (y_2, v_1)\varphi_1(t)]dt \quad \ldots\ldots (35a)$$

$$\& -\int_0^T (y_{2k}, v_2)\varphi_2'(t)dt + \int_0^T [\,(\nabla y_{2k}, \nabla v_2)\varphi_2(t) + (y_{2k}, v_2)\varphi_2(t) + (y_{1k}, v_2)\varphi_2(t)]dt \to -\int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [\,(\nabla y_2, \nabla v_2)\varphi_2(t) + (y_2, v_2)\varphi_2(t) + (y_1, v_2)\varphi_2(t)]dt \quad \ldots\ldots (36a)$$

Since $y_{1k}(0), y_{2k}(0)$ are bounded in $L^2(\Omega)$ and from the Projection theorem, yield

$$(y_{1k}^0, v_1)\varphi_1(0) \to (y_1^0, v_1)\varphi_1(0) \quad \ldots\ldots (35b)$$
$$\& \; (y_{2k}^0, v_2)\varphi_2(0) \to (y_2^0, v_2)\varphi_2(0) \quad \ldots\ldots (36b)$$

Now, to prove that

$$\int_0^T (f_{11}(x,t,y_{1k}), v_1)\varphi_1(t)dt + \int_0^T (\, f_{12}(x,t)u_{1k}, v_1\varphi_1(t))\, dt \to \int_0^T (f_{11}(x,t,y_1), v_1)\varphi_1(t)dt + \int_0^T (\, f_{12}(x,t)u_1, v_1\varphi_1(t))\, dt$$

Let $w_1 = v_1\varphi_1(t)$, $w_1 \in L^\infty(I,V) \subset L^2(I,V) \subset L^2(Q)$, then $w_1(x,t)$ is fixed for fixed $(x,t) \in Q$, let $v_1 \in C[\overline{\Omega}]$, then $w_1 \in C[\overline{Q}]$ is measurable w.r.t. (x,t). let $\bar{f}_{11}(y_{1k}) = f_{11}(y_{1k})w_1$, then $\bar{f}_{11}: Q \times R \to R$ is continuous w.r.t. y_1 for fixed $(x,t) \in Q$ and measurable w.r.t. (x,t) for fixed y_1. Applying Proposition 1.3 in gives the integral $\int_Q f_{11}(y_{1k})w_1\, dxdt$ is continuous w.r.t. y_{1k}, but $y_{1k} \longrightarrow y_1$, strongly in $L^2(Q)$ then $\forall w_1 \in C[\bar{Q}]$, once get

$$\int_Q f_{11}(y_{1k})w_1\, dxdt \to \int_Q f_{11}(y_1)w_1\, dxdt \quad (35c)$$

since $u_{1k} \longrightarrow u_1$,weakly in $L^2(Q)$ then

$\int_Q f_{12}(x,t)u_{1k}w_1 dxdt \to$
$\int_Q f_1(x,t)u_1 w_1\, dxdt$ .... (35d)

The same way can be used to one get that
$\int_Q f_{21}(y_{2k})w_2\, dxdt \to$
$\int_Q f_{21}(y_2)w_2\, dxdt, \forall w_2 \in C[\bar{Q}]$ ....(36c)
$\int_Q f_{12}(x,t)u_{1k}, w_1 dxdt \to$
$\int_Q f_1(x,t)u_1 w_1\, dxdt$ .... (36d)

Finally, using (35a,b,c & d) and (36a,b,c&d) in (35) and (36)respectively, once get
$-\int_0^T (y_1,v_1)\varphi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1)\varphi_1(t)$
$+(y_1,v_1)\varphi_1(t) - (y_2,v_1)\varphi_1(t)]dt$
$= \int_0^T (f_{11}(x,t,y_1),v_1)\varphi_1(t)dt$
$+\int_0^T (\, f_{12}(x,t)u_1,v_1)\varphi_1(t)\, dt$
$+(y_1^0,v_1)\varphi_1(0)$ .... (38)
& $-\int_0^T (y_2,v_2)\varphi_2'(t)dt +$
$\int_0^T [(\nabla y_2,\nabla v_2)\varphi_2(t)$
$+(y_2,v_2)\varphi_2(t) + (y_1,v_2)\varphi_2(t)]dt$
$= \int_0^T (f_{21}(x,t,y_2),v_2)\varphi_2(t)dt$
$+\int_0^T (\, f_{22}(x,t)u_2,v_2)\varphi_2(t)\, dt$
$+(y_2^0,v_2)\varphi_2(0)$ .... (39)
(38) and (39) are hold for each $v_i \in C(\bar{\Omega})$ and since $C(\bar{\Omega})$ is dense in V, then also are hold for every $v_i \in V, \forall i = 1,2$. hence the following two cases are appear:

Case1: Choose $\varphi_i \in D[0,T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0, \forall i = 1,2$. Using integration by parts for the 1^{st} terms in the L.H.S. of (38) and (39), once get $\forall \varphi_1 \in D[0,T]$
$\int_0^T \langle y_{1t},v_1\rangle\varphi_1(t)dt +$
$\int_0^T [(\nabla y_1,\nabla v_1)\varphi_1(t) + (y_1,v_1)\varphi_1(t) -$
$(y_2,v_1)\varphi_1(t)]dt$
$= \int_0^T (f_{11}(x,t,y_1),v_1)\varphi_1(t)dt +$
$\int_0^T \; (f_{12}(x,t)u_1,v_1)\varphi_1(t)\, dt,$ .... (40)
$\Rightarrow$
$\langle y_{1t},v_1\rangle + (\nabla y_1,\nabla v_1) + (y_1,v_1) - (y_2,v_1)$
$= (f_{11}(x,t,y_1),v_1) + (f_{12}(x,t)u_1,v_1),$
$\forall v_1 \in V$, a.e. on I (40a)
i.e.$y_1 = y_{u1}$ satisfies (8a), & $\forall \varphi_2 \in D[0,T]$
$\int_0^T \langle y_{2t},v_2\rangle\varphi_2(t)dt +$
$\int_0^T [(\nabla y_2,\nabla v_2)\varphi_2(t) + (y_2,v_2)\varphi_2(t) +$
$(y_1,v_2)\varphi_2(t)]dt$
$= \int_0^T (f_{21}(x,t,y_2),v_2)\varphi_2(t)dt +$
$\int_0^T \; (f_{22}(x,t)u_2,v_2)\varphi_2(t)\, dt,$ .... (41)
$\Rightarrow$
$\langle y_{2t},v_2\rangle + (\nabla y_2,\nabla v_2) + (y_2,v_2) + (y_1,v_2)$
$= (f_{21}(x,t,y_2),v_2) + (f_{22}(x,t)u_2,v_2)$
$\forall v_2 \in V$, a.e. on I ....(41a)
i.e.$y_2 = y_{u2}$ satisfies (9a).
Case 2: Choose $\varphi_i \in C^1[I]$, such that $\varphi_i(T) = 0$ & $\varphi_i(0) \neq 0, \forall i = 1,2$. Using integration by parts for the 1^{st}terms in the L.H.S. of (40) and (41), one has
$-\int_0^T (y_1,v_1)\varphi_1'(t)dt +$
$\int_0^T [(\nabla y_1,\nabla v_1)\varphi_1(t) + (y_1,v_1)\varphi_1(t) -$
$(y_2,v_1)\varphi_1(t)]dt$
$= \int_0^T (f_{11}(x,t,y_1),v_1)\varphi_1(t)dt +$
$\int_0^T (f_{12}(x,t)u_1,v_1)\varphi_1(t)\, dt +$
$(y_1(0),v_1)\varphi_1(0)$ .... (42)
& $-\int_0^T (y_2,v_2)\varphi_2'(t)dt +$
$\int_0^T [(\nabla y_2,\nabla v_2)\varphi_2(t) + (y_2,v_2)\varphi_2(t) +$
$(y_1,v_2)\varphi_2(t)]dt$
$= \int_0^T (f_{21}(x,t,y_1),v_1)\varphi_1(t)dt +$
$\int_0^T \; (f_{22}(x,t)u_2,v_2)\varphi_2(t)\, dt +$
$(y_2(0),v_2)\varphi_2(0)$ .... (43)

By subtracting (42) from (38) and (43) from (39), one obtain$\forall \varphi_i \in [0,T], \forall i = 1,2$ that
$(y_i^0,v_i)\varphi_i(0) = (y_i(0),v_i)\varphi_i(0), \varphi_i(0) \neq 0$
$\Rightarrow y_i^0 = y_i(0) = y_i^0(x)$.
Thus y_1 & y_2 are solutions of (8-9).
Now, since
$G_1(\vec{u}_k) = \int_Q g_{11}(x,t,y_{1k})\, dxdt +$
$\int_Q g_{12}(x,t,y_{2k})\, dxdt$
Since$\forall i = 1,2$, g_{1i} is independent of u_i and is continuous w.r.t. y_i, then the integral $\int_Q g_{1i}(x,t,y_{ik})\, dxdt$ is continuous w.r.t. y_i, but $\vec{y}_k \longrightarrow \vec{y}$ strongly in $\left(L^2(Q)\right)^2$, then
$\int_Q g_{1i}(x,t,y_{ik})\, dxdt \to \int_Q g_{1i}(x,t,y_i)\, dxdt$
Then $G_1(\vec{u}\,) = \lim_{k\to\infty} G_1(\vec{u}_k) = 0$.
Now, we want to prove that $\forall l = 0,2, G_l(\vec{u}\,)$ is weakly lower semi continuous (W.L.S.C.) w.r.t. $(\vec{y},\vec{u})$. Since $g_{li}(x,t,y_i,u_i)$ is continuous w.r.t. (y_i,u_i) and since $u_i(x,t) \in U_i$ a.e. in Q and U_i is compact, i.e. g_{li} is satisfied the assumptions of lemma 4.2, $\forall i = 1,2$ &$l = 0,2$, which gives
$\int_Q g_{li}(x,t,y_{ik},u_{ik})\; dxdt \to$
$\int_Q g_{li}(x,t,y_i,u_{ik})\; dxdt$ .... (44)

but $g_{li}(x,t,y_i,u_i)$ is convex and continuous w.r.t. u_i then $\int_Q g_{li}(x,t,y_i,u_{ik})\ dxdt$ is also convex and continuous w.r.t. u_i $\Rightarrow$ $\int_Q g_{li}(x,t,y_i,u_i)\ dxdt$ is W.L.S.C. w.r.t. u_i (for each $l=0,2$ & $i=1,2$) i.e.

$\int_Q g_{li}(x,t,y_i,u_i)\ dxdt$
$\leq \lim_{k\to\infty}\inf\int_Q g_{li}(x,t,y_i,u_{ik})\ dxdt$
$=\lim_{k\to\infty}\inf\int_Q (g_{li}(x,t,y_i,u_{ik})-$
$g_{li}(x,t,y_{ik},u_{ik}))dxdt+$
$\lim_{k\to\infty}\inf\int_Q g_{li}(x,t,y_{ik},u_{ik})\ dxdt$.

Then by (44), one obtain that
$\int_Q g_{li}(x,t,y_i,u_i)\ dxdt$
$\leq \lim_{k\to\infty}\inf\int_Q g_{li}(x,t,y_{ik},u_{ik})\ dxdt$

$$\Rightarrow \sum_{i=1}^{2}\int_Q g_{li}(x,t,y_i,u_i)\ dxdt \leq$$

$\lim_{k\to\infty}\inf\sum_{i=1}^{2}\int_Q g_{li}(x,t,y_{ik},u_{ik})\ dxdt$
$\Rightarrow\ G_l(\vec{u})\leq \lim_{k\to\infty}\inf G_l(\vec{u}_k)$, i.e. $G_l(\vec{u})$ is W.L.S.C. w.r.t. $(\vec{y},\vec{u})$, for each $l=0,2$.
but $G_2(\vec{u}_k)\leq 0, \forall k$ then $G_2(\vec{u}\,)\leq 0$, and $G_0(\vec{u})\leq \lim_{k\to\infty}\inf G_0(\vec{u}_k)=\lim_{k\to\infty}G_0(\vec{u}_k)$
$=\inf_{\vec{u}\in\overrightarrow{W}_A} G_0(\vec{\vec{u}}_k)=\min_{\vec{u}\in\overrightarrow{W}_A} G_0(\vec{\vec{u}}_k)$
Which means that $\vec{u}$ is an optimal control.

Assumptions (C):

$g_{l_i y_i}$ and $g_{l_i u_i}$, ($l=0,2$&$i=1,2$) are of Carathéodory type (or continuous) on $Q\times(R\times R)$ and are satisfied $\forall(x,t)\in Q$, and $y_i,u_i\in R$

$\left|g_{l_i y_i}(x,t,y_i,u_i)\right|\leq \eta_{l_{i5}}(x,t)+c_{l_{i5}}|y_i|$
$+\acute{c}_{l_{i5}}|u_i|, \eta_{l_{i5}}\in L^2(Q)$
$\left|g_{l_i u_i}(x,t,y_i,u_i)\right|\leq \eta_{l_{i6}}(x,t)+c_{l_{i6}}|y_i|+$
$+\acute{c}_{l_{i6}}|u_i|, \eta_{l_{i6}}\in L^2(Q)$.

<u>Theorem 4.2:</u>

Dropping the index l in g_l & G_l. With assumptions (A), (B) and (C), the following adjoint $(z_1,z_2)=(z_{u1},z_{u2})$ equations are given by

$-z_{1t}-\Delta z_1+z_1+z_2$
$=z_1 f_{y1}(x,t,y_1,u_1)+g_{y1}(x,t,y_1,u_1)$ $-z_{2t}-\Delta z_2+z_2-z_1$
$=z_2 f_{y2}(x,t,y_2,u_2)+g_{y2}(x,t,y_2,u_2)$
$z_1(T)=0$ and $z_2(T)=0$, on Γ

And the Hamiltonian is defined:
$H(x,t,y_i,z_i,u_i)=$
$=\sum_{i=1}^{2}(z_i f_i(x,t,y_i,u_i)+g_i(x,t,y_i,u_i))$
Then the Fréchet derivative of G is given by
$$\acute{G}(\vec{u})\cdot\overrightarrow{\delta u}=\int_Q \begin{pmatrix} z_1 f_{u1}+g_{u1}\\ z_2 f_{u2}+g_{u2}\end{pmatrix}\cdot\begin{pmatrix}\delta u_1\\ \delta u_2\end{pmatrix}dxdt$$

***Proof*:**

At first let
$G(\vec{u})=\int_Q g_1(x,t,y_1,u_1)dxdt+$
$\int_Q g_2(x,t,y_2,u_2)dxdt$
Where $\vec{u}$ is a given control and $\vec{y}$, is its corresponding solution of the state equation.
From the assumptions on g_1 and g_2, the definition of the Fréchet derivative, the result of Lemma 3.1, and then using Minkowiski inequality, we have
$G_0(\vec{u}+\delta\vec{u})-G_0(\vec{u})$
$=\int_\Omega\left(g_{1y_1}\delta y_1+g_{1u_1}\delta u_1\right)dx+$
$\int_\Omega\left(g_{2y_2}\delta y_2+g_{2u_2}\delta u_2\right)dx$
$+\varepsilon_1\left(\overrightarrow{\delta u}\right)\left\|\overrightarrow{\delta u}\right\|_0$ .... (45)
where $\varepsilon_1\left(\overrightarrow{\delta u}\right)\longrightarrow 0$& $\left\|\overrightarrow{\delta u}\right\|_0\longrightarrow 0$ as $\overrightarrow{\delta u}\longrightarrow 0$.

On the other hand, the weak forms of the adjoint equations are
$-\langle z_{1t},v_1\rangle+(\nabla z_1,\nabla v_1)+(z_1,v_1)$
$+(z_2,v_1)=\left(z_1 f_{1y1},v_1\right)+\left(g_{1y1},v_1\right)$ ..... (46)
& $-\langle z_{2t},v_2\rangle+(\nabla z_2,\nabla v_2)+(z_2,v_2)$
$-(z_1,v_2)=\left(z_2 f_{2y2},v_2\right)+\left(g_{2y2},v_2\right)$ .... (47)

The proof of the unique solution of the weak form (46-47) is similar to the proof of the unique solution of the state equation (8-9). Substituting $v_1=\delta y_1$in (46) and $v_2=\delta y_2$in (47), integrating both sides from 0 to T and then integration by parts for the 1^{st} terms of each obtained equation, one has
$\int_0^T\langle\delta y_{1t},z_1\rangle\,dt+\int_0^T[\,(\nabla z_1,\nabla\delta y_1)+$
$(z_1,\delta y_1)+(z_2,\delta y_1)]dt=$
$\int_0^T[\,\left(z_1 f_{1y1},\delta y_1\right)+\left(g_{1y1},\delta y_1\right)]dt$ .... (48)
& $\int_0^T\langle\delta y_{2t},z_2\rangle\,dt+\int_0^T[\,(\nabla z_2,\nabla\delta y_2)+$
$(z_2,\delta y_2)-(z_1,\delta y_2)]dt=$
$\int_0^T[\,\left(z_2 f_{2y2},\delta y_2\right)+\left(g_{2y2},\delta y_2\right)]dt$ .... (49)

Substituting the solution $y_1$once in (12) and then the solution $y_1+\delta y_1$ once again, subtracting the obtained equations one from the other, with $v_1=z_1$, we have

$\int_0^T\langle\delta y_{1t},z_1\rangle\,dt+\int_0^T[\,(\nabla\delta y_1,\nabla z_1)+(\delta y_1,z_1)-(\delta y_2,z_1)]dt=\int_0^T(f_1(y_1+\delta y_1,u_1+\delta u_1),z_1)dt-\int_0^T(f_1(y_1,u_1),z_1)dt$ .... (50)

Also substituting the solutions $y_2$once in (13) and then the solution $y_2+\delta y_2$ once again, subtracting the obtained equations one from the other, with$v_2=z_2$, we have

$\int_0^T\langle\delta y_{2t},z_2\rangle\,dt+\int_0^T[\,(\nabla\delta y_2,\nabla z_2)+(\delta y_2,z_2)+(\delta y_1,z_2)]dt=\int_0^T(f_2(y_2+\delta y_2,u_2+\delta u_2),z_2)dt-\int_0^T(f_2(y_2,u_2),z_2)dt$ .... (51)

From the assumptions on f_1 and f_2, the Fréchet derivatives of f_1 and f_2 are exist, then from the result of Lemma 3.1 and the Minkowiski inequality, once get

$\int_0^T\langle\delta y_{1t},z_1\rangle\,dt+\int_0^T[\,(\nabla\delta y_1,\nabla z_1)+(\delta y_1,z_1)-(\delta y_2,z_1)]dt=\int_0^T(\,f_{1y1}\delta y_1+f_{1u1}\delta u_1,z_1)dt+\varepsilon_2(\overrightarrow{\delta u})\|\overrightarrow{\delta u}\|_Q$ .... (52)

&

$\int_0^T\langle\delta y_{2t},z_2\rangle\,dt+\int_0^T[\,(\nabla\delta y_2,\nabla z_2)+(\delta y_2,z_2)+(\delta y_1,z_2)]dt=\int_0^T(\,f_{2y2}\delta y_2+f_{2u2}\delta u_2,z_2)dt+\varepsilon_3(\overrightarrow{\delta u})\|\overrightarrow{\delta u}\|_Q$ .... (53)

Subtracting (52) and (53) from (48) and (49), adding the obtain equations, one get

$\int_0^T[\,(f_{1u1}\delta u_1,z_1)+(f_{2u2}\delta u_2,z_2)]dt+\varepsilon_4(\overrightarrow{\delta u})\|\overrightarrow{\delta u}\|_Q=\int_0^T[\,(g_{1y1},\delta y_1)+(g_{2y2},\delta y_2)]dt$ (54)

Where $\varepsilon_4(\overrightarrow{\delta u})=\varepsilon_2(\overrightarrow{\delta u})+\varepsilon_3(\overrightarrow{\delta u})\longrightarrow 0$, as $\|\overrightarrow{\delta u}\,\|_Q\longrightarrow 0$

Now, substituting (54) in (45), once get

$G(u_1+\delta u_1)-G(u_1)=\int_Q(z_1f_{1u1}+g_{1u1})\delta u_1\,dxdt+\int_Q(z_2f_{2u2}+g_{2u2})\delta u_2\,dxdt+\varepsilon_5(\overrightarrow{\delta u})\|\overrightarrow{\delta u}\,\|_Q$ .... (55)

Where $\varepsilon_5(\overrightarrow{\delta u})=\varepsilon_1(\overrightarrow{\delta u})+\varepsilon_4(\overrightarrow{\delta u})\longrightarrow 0$, as $\|\overrightarrow{\delta u}\,\|_Q\longrightarrow 0$

Hence the Fréchet derivative of G is

$\acute{G}(\vec{u})\cdot\overrightarrow{\delta u}=\int_Q\begin{pmatrix}z_1f_{1u1}+g_{1u1}\\ z_2f_{2u2}+g_{2u2}\end{pmatrix}\cdot\begin{pmatrix}\delta u_1\\ \delta u_2\end{pmatrix}dxdt$

5. Necessary and sufficient conditions for optimality:

In this section the necessary theorem for optimality under prescribed assumptions is proved so as the sufficient theorem for optimality as follows:

Theorem 5.1: ***Necessary Conditions for Optimality (Multipliers Theorem):***

a) with assumptions (A),(B) and (C) if $\overrightarrow{W}$ is convex, the control $\vec{u}\in\overrightarrow{W_A}$ is optimal, then there exist multipliers $\lambda_l\in\mathbb{R}$, $l=0,1,2$ with $\lambda_0\geq 0,\lambda_2\geq 0,\sum_{l=0}^{2}|\lambda_l|=1$ such that the following Kuhn-Tucker-Lagrange (K.T.L.) conditions are satisfied: $\int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\cdot\overrightarrow{\delta u}dxdt\geq 0$

$\forall\vec{w}\in\overrightarrow{W},\overrightarrow{\delta u}=\vec{w}-\vec{u}$ ....(56a)

where $g_i=\sum_{l=0}^{2}\lambda_l g_{li}$ and $z_i=\sum_{l=0}^{2}\lambda_l z_{li}$ $(i=1,2)$ in the definition of H and $\vec{z}$, and also the Transversality condition is

$\lambda_2G_2(\vec{u})=0$ .... (56b)

(b)(Minimum principle in weak form) If $\overrightarrow{W}=\overrightarrow{W_{\vec{U}}}$ then inequality (56a) is equivalent to the minimum principle in point wise form

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}).\vec{u}(t)=\min_{\vec{w}\in\vec{U}}H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}).\vec{w}$, a. e. on$Q$, ... (57)

***Proof*:**

a) From Theorem(4.2)we get that the functional $G_l(\vec{u})$ has a continuous Fréchet derivative at each $\vec{u}\in\overrightarrow{W}$, since the control $\vec{u}\in\overrightarrow{W_A}$ is optimal, then using the K.T.L. theorem there exist multipliers $\lambda_l\in\mathbb{R}$, $l=0,1,2$ with $\lambda_0\geq 0,\lambda_2\geq 0,\sum_{l=0}^{2}|\lambda_l|=1$,such that $\forall\vec{w}\in\overrightarrow{W}$

$\left(\lambda_0\vec{G}_{0\vec{u}}(\vec{u})+\lambda_1\vec{G}_{1\vec{u}}(\vec{u})+\lambda_2\vec{G}_{2\vec{u}}(\vec{u})\right).(\vec{w}-\vec{u})\geq 0$

and $\lambda_2G_2(\vec{u})=0$

Substituting the Fréchet derivatives of $G_l(\vec{u})$ (for $l=0,1,2$) in the above inequality, i.e. $\sum_{i=1}^{2}\int_Q[(z_if_{iui}+g_{iui})]\,\delta u_i dxdt\geq 0$, where $g_i = \sum_{l=0}^{2} \lambda_l\, g_{li},\ z_i = \sum_{l=0}^{2} \lambda_l\, z_{li},\ \forall i =$ 1,2, or

$\int_\Omega (z_1 f_{1u_1} + g_{1u_1}, z_2 h_{1u_2} + g_{2u_2}).\overrightarrow{\delta u}\, dx \geq 0$

$\Rightarrow \int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}) \cdot \overrightarrow{\delta u} dxdt \geq 0,$

$\forall \vec{w} \in \vec{W}, \overrightarrow{\delta u} = \vec{w} - \vec{u}$

To prove the second part, let $\{\vec{w}_k\}$ dense in a set $\vec{W}$, μ is Lebesgue measure on Q and let $S \subset Q$ be a measurable set such that

$$\vec{w}(x,t) = \begin{cases} \vec{w}_k(x,t), if\ (x,t) \in S \\ \vec{u}(x,t), if\ (x,t) \notin S \end{cases}$$

Therefore (56a) becomes for each S

$$\int_S H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\,(\vec{w}_k - \vec{u}) \geq 0\ \Rightarrow$$

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})(\vec{w}_k - \vec{u}) \geq 0$,a.e.in Q ...... (58)

This inequality holds in a set $P = \bigcap_k P_k$, where $P_k = Q - Q_k$, $\mu(Q_k) = 0$, $\forall k$, but P is independent of k with $\mu(Q-P) = 0$

but $\{\vec{w}_k\}$ is dense in $\vec{W}$, then (58) becomes

$H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})(\vec{w} - \vec{u}) \geq 0$, a.e. in Q $\Rightarrow$ $H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{u} = \min_{\vec{w}\in\vec{U}} H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\vec{w}$, a.e. in Q.

The converse is clear.

Theorem 5.2: (Sufficient Conditions for Optimality): In Addition to the assumptions (A), (B) and (C), Suppose that $\vec{W}$ is convex, f_1, f_2 and $g_{1i}$are affine w.r.t. (y_i, u_i) for each$(x,t)$and g_{0i}&$g_{2i}$are convex w.r.t. (y_i, u_i) for each (x,t),$\forall i = 1,2$. Then the necessary conditions in Theorem 5.1 with $\lambda_0 > 0$ are also sufficient.

***Proof*:**

Assume $\vec{u}$ is satisfied the K.T.L. condition, and $\vec{u} \in \vec{W}_A$, i.e.

$\int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\overrightarrow{\delta u}\, dxdt \geq 0,\ \forall \vec{w} \in \vec{W}$ and $\lambda_2 G_2(\vec{u}) = 0$

Let $G(\vec{u}) = \sum_{l=0}^{2} \lambda_l G_l(\vec{u})$, then using theorem 4.2, we have

$\acute{G}(\vec{u}) \cdot \overrightarrow{\delta u} =$

$\sum_{l=0}^{2} \lambda_l \int_Q \sum_{i=1}^{2} (z_{li} f_{iui} + g_{liui})\, \delta u_i\, dxdt$

$= \int_Q H_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u})\overrightarrow{\delta u}\, dxdt \geq 0$

Since

$f_1(x,t,y_1,u_1) = f_{11}(x,t)y_1 + f_{12}(x,t)u_1$
$+f_{13}(x,t)$, and
$f_2(x,t,y_2,u_2) = f_{21}(x,t)y_2 + f_{22}(x,t)u_2$
$+f_{23}(x,t)$,

Let $\vec{u} = (u_1,u_2)$ &$\vec{\bar{u}} = (\bar{u}_1,\bar{u}_2)$ are two given controls vectors, then $\vec{y} = (y_{u1}, y_{u2}) = (y_1,y_2)$ & $\vec{\bar{y}} = (\bar{y}_{\bar{u}1}, \bar{y}_{\bar{u}2}) = (\bar{y}_1,\bar{y}_2)$ are their corresponding stats solutions. Substituting the pair $(\vec{u},\vec{y})$ in equations (1-6) and multiplying all the obtained equations by $\alpha \in [0,1]$ once and then substituting the pair $(\vec{\bar{u}},\vec{\bar{y}})$ in (1-6) once again and multiplying all the obtained equations by $(1-\ \alpha)$, finally adding each pair from the corresponding equations together one gets:

$(\alpha y_1 + (1-\alpha)\bar{y}_1\,)_t - \Delta(\alpha y_1 + (1-\alpha)\bar{y}_1) +$
$(\alpha y_1 + (1-\alpha)\bar{y}_1) - (\alpha y_2 + (1-\alpha)\bar{y}_2)$
$= f_{11}(x,t)(\alpha y_1 + (1-\alpha)\bar{y}_1) +$
$f_{12}(x,t)(\alpha u_1 + (1-\alpha)\bar{u}_1) + f_{13}(x,t)$
.................................(59a)

$\alpha y_1(x,t) + (1-\alpha)\bar{y}_1(x,0) = 0$ .... (59b)

$\alpha y_1(x,0) + (1-\alpha)\bar{y}_1(x,0) = y_1^0(x)$ ....(59c)

$(\alpha y_2 + (1-\alpha)\bar{y}_2\,)_t - \Delta(\alpha y_2 +$
$(1-\alpha)\bar{y}_2) + (\alpha y_2 + (1-\alpha)\bar{y}_2) + \alpha$
$(\alpha y_1 + (1-\alpha)\bar{y}_1)$
$= f_{21}(x,t)(\alpha y_2 + (1-\alpha)\bar{y}_2) +$
$f_{22}(x,t)(\alpha u_2 + (1-\alpha)\bar{u}_2) + f_{23}(x,t)$
.................................(60a)

$\alpha y_2(x,t) + (1-\alpha)\bar{y}_2(x,0) = 0$ .... (60b)

$\alpha y_2(x,0) + (1-\alpha)\bar{y}_2(x,0) = y_2^0(x)$ ....(60c)

Equations (59) and (60) tell us that if we have the control vector $\vec{\tilde{u}} = (\tilde{u}_1,\tilde{u}_2)$ with $\vec{\tilde{u}} = \alpha\vec{u} + (1-\alpha)\vec{\bar{u}}$ then its corresponding state vector (solution) is $\vec{\tilde{y}} = (\tilde{y}_1,\tilde{y}_2)$ with $\tilde{y}_i = y_{i\tilde{u}_i} = y_{i(\alpha u_i + (1-\ \alpha)\bar{u}_i)} = \alpha y_i + (1-\ \alpha)\bar{y}_i$, for each $i = 1,2$. So we get the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ is convex – linear w.r.t. $(\vec{y},\vec{u})$ for each $(x,t)) \in Q$.

On the other hand, the function $G_1(\vec{u})$ is convex – linear w.r.t. $(\vec{y},\vec{u})$, $\forall (x,t) \in Q$, this back to the fact that the sum of two affine functions $g_{1i}(x,t,y_i,u_i)$ $(\forall\, i = 1,2)$ w.r.t. (y_i,u_i) and $\forall (x,t) \in Q$ is affine and the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ is convex-linear.

Also, since the functions $G_0(\vec{u})$ and $G_2(\vec{u})$ are convex w.r.t. $(\vec{y},\vec{u})$, $\forall (x,t) \in Q$ (from the assumptions on the functions g_{l1} and g_{l2}, l =0,2 and from the fact that the sum of two integral of convex function is also convex). then $G(\vec{u})$ is convex w.r.t. $(\vec{y},\vec{u})$, $\forall (x,t) \in Q$

in the convex set $\overrightarrow{W}$, and has a continuous Fréchet derivative satisfies

$\acute{G}(\vec{u})\overrightarrow{\delta u} \geq 0 \implies G(\vec{u})$ has a minimum at $\vec{u} \implies G(\vec{u}) \leq G(\vec{w}), \forall \vec{w} \in \overrightarrow{W} \implies \lambda_0 G_0(\vec{u}) + \lambda_1 G_1(\vec{u}) + \lambda_2 G_2(\vec{u}) \leq \lambda_0 G_0(\vec{w}) + \lambda_1 G_1(\vec{w}) + \lambda_2 G_2(\vec{w})$

Let $\vec{w} \in \overrightarrow{W}_A$, with $\lambda_2 \geq 0$ and from the Transversality condition, we get
$\lambda_0 G_0(\vec{u}) \leq \lambda_0 G_0(\vec{w}), \forall \vec{w} \in \overrightarrow{W} \Rightarrow$
$G_0(\vec{u}) \leq G_0(\vec{w}), \forall \vec{w} \in \overrightarrow{W}$, since $(\lambda_0 > 0)$
$\therefore \vec{u}$ is an optimal control for the problem.

6. Conclusions

The Galerkin method with the compactness theorem are used successfully to prove the existence and the uniqueness "continuous state vector" solution for a couple nonlinear parabolic partial differential equations for fixed continuous classical control vector. The existence theorem of a continuous classical optimal control vector governing by the considered couple of nonlinear partial differential equation of parabolic type with equality and inequality constraints is proved. The existence and the uniqueness solution of the couple of adjoint equations associated with the considered couple equations of the state is studied. The Frcéhet derivation of the Hamiltonian is derived. The necessary conditions theorem so as the sufficient conditions theorem of optimality of the constrained problem are developed and proved.

References

[1] Aderinto, Y. O. & Bamigbola, M.O.(2012). A qualitative study of the optimal control model for an electric power generating system. *Journal of Energy in Southern Africa*, Vol 23, No 2, pp.65-72.

[2] Agusto, F. B.& Bamigbola, O.M. (2007). Optimal Control for Bacterial Transport. *AJST*, Vol. 8, No. 2, pp. 18-23.

[3] Al-Hawasy J. (2008). The Continuous Classical Optimal Control of a Nonlinear Hyperbolic Equation (CCOCP). *Al-Mustansiriyah Journal of Science,* Vol.19 No.8 pp.96-110.

[4] Al-Hawasy J. and Al-Rawdhanee E. (2014),” The Continuous Classical Optimal Control of a Coupled of Non-linear Elliptic Equations”, Mathematical Theory and Modeling, Vol.4, No.14, pp.211-219.

[5] Amini F. & Afshar M. Amin (2008). Modified Predictive optimal linear Control of Structures in Seismic Region. *Iranian Journal of Science & Technology, Transaction B, Engineering*, Vol. 32, No. B2, pp. 91-106.

[6] Bors D., S.Walczak, (2005). Optimal control elliptic systems with distributed and boundary controls. *Nonlinear Analysis* 63, e1367 – e1376.

[7] Borzabadi, A. H., Kamyad, A. V. & Farahi, M. H, G.(2004). Optimal Control of the Heat Equation in an Inhomogeneous Body. *J. Appl. Math. and Computing*, Vol. 15, No. 1 - 2, pp. 127 – 146.

[8] Boucekkine, A., Camacho, C. & Fabbri, G.(2013).On the Optimal Control of Some Parabolic Partial Differential Equations Arising in Economics. *Serdica Math. J*,Vol 39, pp.331–354.

[9] Brauer, F. and Nohel J. A., “Ordinary Differential Equations: A First Course”, 2nd edition, W.A. Benjamin, 1973.

[10] Budigono, A. & Wibowo, S.S. (2007). Optimal Tracking Controller Design for a Small Scale Helicopter. *ICIUS*, Oct 24-25 Bali, Indonesia.

[11] Chryssoverghi I. & Al-Hawasy J., The Continuous Classical Optimal Control Problem of a Semilinear parabolic Equations (CCOCP)(2010). Journal of Kerbala University, Vol. 8 No.3, pp. 57-70.

[12] Chryssoverghi I. (2003), Optimization, *National Technical University of Athens, Athens-Grecce.*

[13] El hiaet, M., Balatif, O., Bouyaghroumni, J., Labriji, E. & Rachik, M.(2012). Optimal Control Applied to the Spread of Influenza A(H1N1). *Applied Mathematical Sciences*, Vol. 6 No. 82, PP. 4057 – 4065.

[14] Lions, J. L. (1972) Optimal control of systems governed by partial differential equations. Springer-Verlag, New York.

[15] Orpel, A.(2009). Optimal Control Problems with Higher Order Constraints. *Folia Mathematica*, Vol 16, No. 1, pp. 31–44.

[16] Rubio, J. de J., Torres C. & Aguilar C.(2011). Optimal Control Based in A mathematical model Applied to Robotic

Arms. *International Journal of Innovative Computing, Information and Control*, Vol. 7, No.8. pp. 5045-5062.
[17] Temam, R. (1977). Navier-Stokes Equations. North-Holand Publishing Company.
[18] Warga, J.(1972). Optimal Control of Differential and Functional Equations. Academic Press: New York and London.

الخلاصة

يهتم هذا البحث بمسألة وجود ووحدانية حل لزوج من المعادلات التفاضلية من النوع المكافئ باستخدام طريقة كاليركن عندما يكون متجه السيطرة التقليدية "Classical control vector" ثابتا". يتناول ايضا" برهان لمبرهنة وجود سيطرة امثلية مستمرة تقليدية بوجود قيدي التساوي وعدم التساوي. كذلك برهان مبرهنة وجود حل لزوج من المعادلات الملحقة "Adjoint equations" لمعادلات الحالة. تم اشتقاق مشتقة فريشيه "Frcéhet" لدالة هاملتون الخاصة بهذه المسالة. ايضا تم برهان مبرهنتا الشروط الضرورية والكافية لوجود متجه سيطرة امثلية مستمرة تقليدية بوجود قيدي التساوي وعدم التساوي.