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Solution of Coupled System of Caputo Fractional Differential Equations with Multi-Point Boundary Conditions

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Abstract

This article relies on the Caputo fractional derivative for the objective of is to examine an interconnected system of fractional differential equations. The problem under consideration involves four fractional-Caputo-derivatives underneath initial conditions. We state and prove the existence and uniqueness theorem of solution by application of Banach fixed point theorem. Then, another result that deals with the existence of at least one solution is delivered and some sufficient conditions for this result are established by means of the fixed-point theorem of Schaefer. We end the paper by presenting to the reader by illustrative example.

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1. Introduction

The study of fractional calculus and differential equations of fractional order is highly significant due to their applicability in the analysis and simulation of real-world phenomena. Significant advancements have been made in the research of fractional-order differential equations [1]. To find other research works with practical applications, readers can refer to the provided sources [2]. Research on the presence of unique solutions for fractional differential equations is crucial as [3] it enhances physicians' comprehension of the dynamics of real phenomena. Refer to the papers [4,5], for additional information. Furthermore, they have great significance in different science topics [6]. The Lane-Emden equation is an equation in astrophysics that holds great significance [7,8]. For further information, please refer to the sources. [9-12], and the corresponding references. The reader can also consult, [13-16], for more details about fractional equations, we study existence and uniqueness of the solution of the following system [17]:

$$\begin{cases} D^{\alpha}u(t) = \mathcal{F}\left(t, v^{\delta}(t), D^{\delta}v(t)\right) \\ D^{\beta}u(t) = \mathcal{F}\left(t, u^{\sigma}(t), D^{\delta}u(t)\right) \\ \quad 1 < \alpha, \beta < 2, \;\; t \in [0,1] \end{cases} \quad \dots (1)$$

Where $\sigma \le \alpha - 1, \delta \le \beta - 1$, D is the Caputo fractional derivative. We denote by J = [0, 1], [18]. The structure of this document is as follows:
In Section two, we use some definitions and definitions. To support our main findings, In the third section, we prove the main theorems of this paper, discussing examples. To our knowledge, there are no papers that address this problem with Caputo fractional.

2. Preliminaries

For the convenience of the reader, we present some concepts of fractional calculus.

Definition 2.1 (Riemann-Liouville (R–L) type fractional order integral), [19]. The R-L fractional integral of order $\alpha > 0$ is define as follows:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \quad \dots (2)$$

Γ represents the Gamma function

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Definition 2.2, [20]. If $f \in C^n([0,1], R)$ and $n-1 < \alpha \le n$, then, the Caputo fractional derivative is:

$$D^\alpha f(t) = I^{n-\alpha}\frac{d^n}{dt^n}\left(f(t)\right) \quad \dots (3)$$
$$= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds$$

Lemma 2.3. [19] Taking $n \in N$ and

$$n-1 < \alpha \le n, \quad y(t) = \sum_{j=0}^{n-1} c_i t^I \quad \dots (4)$$

Then Eq. (4) is general solution of $D^\alpha y(t) = 0$. such that $c_j \in \mathbb{R}, i = 0,1,2,..,n-1$.

Lemma 2.4 [10] Taking $n \in N^*$ and $n-1 < \alpha < n$, and from eq. (4) then, we have

$$I^\alpha D^\alpha y(t) = y(t) + \sum_{j=0}^{n-1} c_j t^i \quad \dots (5)$$

such that $c_j \in \mathbb{R}, i = 0,1,2,..,n-1$.

Lemma 2.5. Let $h \in C([0,1])$. The problem is then stated:

$$\begin{cases} D^\alpha u(t) = h(t), \quad 1 < \alpha \le 2, \; t \in [0,1] \\ u(0) = 0, \; u(1) = \sum_{i=1}^{k} a_i I^p u(b_i) \end{cases} \quad \dots (6)$$

that the following representation is an integral solution:

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h(\tau)\, d\tau + \frac{t}{\Delta}\Big(\frac{\sum_{i=1}^k a_i}{\Gamma(\alpha+p)}\int_0^t (b_i-\tau)^{\alpha+p-1} h(\tau)\, d\tau - \int_0^t (1-\tau)^{\alpha-1} h(\tau)\, d\tau \quad \dots (7)$$

where $\Delta = \frac{\sum_{i=1}^k \boldsymbol{b_i^{p+1}}}{\boldsymbol{\Gamma(\alpha+1)}}$

Proof.

Using the result of Lemma 2.4 and applying it to equation (6), we find that

$$u(t) = I^\alpha h(t) + c_0 - c_1 \quad \dots (8)$$

By applying the initial conditions, we are able to express:

$u(0) = 0 \Rightarrow c_0 = 0$ and

$$\frac{1}{\Gamma(\alpha)}\int_0^t (1-\tau)^{\alpha-1} h(\tau)\, d\tau - c_1 = \sum_{i=1}^k a_i I^p (I^\alpha h(b_i) + c_1 b_i)$$

$$= \sum_{i=1}^k a_i I^p (I^\alpha h(b_i) + c_1 b_i)$$

$$c_1\Big(1 - \frac{1}{\Gamma(p)}\sum_{i=1}^k a_i \int_0^t (b_i-\tau)^{p-1} h(\tau) d\tau$$

$$= \frac{1}{\Gamma(p+\alpha)}\sum_{i=1}^k a_i \int_0^t (b_i-\tau)^{\alpha+p-1} h(\tau) d\tau$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h(\tau)\, d\tau$$

Replacing $\boldsymbol{c_1}, \boldsymbol{c_0}$. in (8), we end the proof.

3. Existence and Uniqueness Results

Let's transform the above equation (8) into a fixed-point problem. First, let us start by examining the Banach space.

$(E,F) = \{u, v \in \mathbb{C}(J,\mathbb{R}), D^\delta u, D^\delta v \in \mathbb{C}(J,\mathbb{R})\}$

endowed with the norm

$\| E,F \|_E = \| u + D^\delta u \|_\infty + \| v + D^\delta v \|_\infty$

such that

$\| u \|_\infty = \sup_{t\in J} |u(t)|, \| D^\delta u \|_\infty = \sup_{t\in J} |D^\delta u(t)|$.

Now considered:

(A1). The functions $f, g : J \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

(A2) There exists a nonnegative continuous functions $\lambda_i, v_i \in J, i = 1,2$ such that for all $t \in J$ and $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$, we have

$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le \lambda_1(t)|u_1 - u_2| + v_1(t)|v_1 - v_2|$,

$|g(t,u_1,v_1) - f(t,u_2,v_2)| \le \lambda_2(t)|u_1 - u_2| + v_2(t)|v_1 - v_2|$

(A3) There exists a nonnegative functions $m_1(t)$ and $m_2(t)$ such that

$|f(t,u,v)| \le m_1(t)$, $|g(t,u,v)| \le m_2(t)$,

$\forall t \in J, u, v \in \mathbb{R}$, with

$L_f, L_g = \sup_{t\in J}(m_1, m_2)(t)$, We take

$$D_1 = \frac{2}{\Gamma(\alpha+1)} + \frac{\sum_{i=1}^k a_i\; b_i^{\alpha+p}}{|\Delta|\Gamma(\alpha+p+1)}$$
$$D_2 = \frac{2}{\Gamma(\alpha-\sigma+1)} + \frac{\sum_{i=1}^k a_i\; b_i^{\alpha+p}}{|\Delta|\Gamma(2-\sigma)\Gamma(\alpha+p+1)}$$
$$D_3 = \frac{2}{\Gamma(\beta+1)} + \frac{\sum_{i=1}^k c_i\; d_i^{\beta+q}}{|\bar{\Delta}|\Gamma(\beta+q+1)}$$
$$D_4 = \frac{2}{\Gamma(\beta-\delta+1)} + \frac{\sum_{i=1}^k c_i\; d_i^{\alpha+p}}{|\bar{\Delta}|\Gamma(2-\sigma)\Gamma(\alpha+p+1)}$$

$$\Delta = 1 - \frac{\sum_{i=1}^k a_i\, b_i^{p+1}}{\Gamma(\alpha+1)}, \; \bar{\Delta} = 1 - \frac{\sum_{i=1}^k c_i\, d_i^{p+1}}{\Gamma(\beta+1)}$$

3.1 Theorem: If (A1) and (A2) are satisfied and

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$(D_1 + D_2)(\theta_1 + \nu_1) + (D_3 + D_4)(\theta_2 + \nu_2) < 1.$
Then (1) there is a unique solution for the interval J.

Proof. Let us consider the operator:
$T : E \times F \longrightarrow E \times F$ defined by:

$$T(u,v)(t) = \big(T_1 v(t),\, T_2 u(t)\big), t \in J, \quad \dots(9)$$

where

$$Tv(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f\left(\tau, v(\tau), D^\delta v(\tau)\right) d\tau$$
$$+\frac{t}{\Delta}\Big(\frac{\sum_{i=1}^k a_i}{\Gamma(\alpha+p)}\int_0^{b_i} (b_i-\tau)^{\alpha+p-1} f\left(\tau, v(\tau), D^\delta v(\tau)\right) d\tau$$
$$-\frac{t}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f\left(\tau, v(\tau), D^\delta v(\tau)\right) d\tau$$

and

$$Tu(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} g\big(\tau, v(\tau), D^\sigma v(\tau)\big) d\tau$$
$$+\frac{t}{\Delta}\Big(\frac{\sum_{i=1}^k c_i}{\Gamma(\beta+q)}\int_0^{d_i} (d_i-\tau)^{\beta+p-1} f\big(\tau, v(\tau), D^\sigma v(\tau)\big) d\tau$$
$$-\frac{t}{\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1} f\big(\tau, v(\tau), D^\sigma v(\tau)\big) d\tau$$

Now, to show that T is a contraction mapping.
Let $(u,v),(u^*,v^*) \in E \times F$. Then for all $t \in J$, we have:

$$|T_1 v(t) - T_1 v^*(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Big| f\left(\tau, v(\tau), D^\delta v(\tau)\right) - f\left(\tau, v(\tau), D^\delta v^*(\tau)\right)\Big| d\tau$$
$$+\frac{t}{|\Delta|}\Big(\frac{\sum_{i=1}^k a_i}{\Gamma(\alpha+p)}\int_0^{b_i} (b_i-\tau)^{\alpha+p-1}\Big| f\left(\tau, v(\tau), D^\delta v(\tau)\right) - f\left(\tau, v(\tau), D^\delta v^*(\tau)\right)\Big| d\tau +$$
$$\frac{t}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\Big| g\left(\tau, v(\tau), D^\delta v(\tau)\right) - g\left(\tau, v(\tau), D^\delta v^*(\tau)\right)\Big| d\tau$$

Using the hypothesis (A2) we get

$$|T_1 v(t) - T_1 v^*(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (1-\tau)^{\alpha-1}(\theta_1\|v-v^*\| + \nu_1\|D^\delta v - D^\delta v^*\|) d\tau$$
$$+\frac{t}{|\Delta|}\Big(\frac{\sum_{i=1}^k a_i}{\Gamma(\alpha+p)}\int_0^{b_i} (b_i-\tau)^{\alpha+p-1}\big(\theta_1\|v-v^*\| + \nu_1\|D^\delta v - D^\delta v^*\|\big) d\tau$$
$$+\frac{t}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}(\theta_1\|v-v^*\| + \nu_1\|D^\delta v - D^\delta v^*\|) d\tau$$

Consequently, we have,

$$|T_1 v(t) - T_1 v^*(t)| \le \Big(\frac{2}{\Gamma(\alpha+p)} + \frac{\sum_{i=1}^k a_i\ b_i^{\alpha+p}}{|\Delta|\Gamma(\alpha+p+1)}(\theta_1\|v-v^*\| + \nu_1\|D^\delta v - D^\delta v^*\|)$$

which implies that

$$|T_1 v(t) - T_1 v^*(t)| \le \Big(\frac{2}{\Gamma(\alpha+1)} + \frac{\sum_{i=1}^k a_i\ b_i^{\alpha+p}}{|\Delta|\Gamma(\alpha+p+1)}(\theta_1+\nu_1)\|v-v^*\|$$

$+\|D^\delta v - D^\delta v^*\|)$ And then we have

$$|T_1 v(t) - T_1 v^*(t)| \le \mathrm{D}_1(\theta_1+\nu_1)(\|v-v^*\| + \|D^\delta v - D^\delta v^*\|)$$

We have also

$$|D^\sigma T_1 v(t) - D^\sigma T_1 v^*(t)| \le \frac{1}{\Gamma(\alpha-\sigma)}\int_0^t (1-\tau)^{\alpha-\sigma-1}\big| f\big(\tau, v(\tau), D^\sigma v(\tau)\big) - g\big(\tau, v^*(\tau), D^\sigma v^*(\tau)\big)\big| d\tau$$
$$+\frac{t^{1-\sigma}}{\Gamma(2-\sigma)|\Delta|}\Big(\frac{\sum_{i=1}^k a_i}{\Gamma(\alpha+p)}\int_0^{b_i} (b_i-\tau)^{\alpha+p-1}$$

By (A2), we have

$$|D^\sigma T_1 v(t) - D^\sigma T_1 v^*(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (1-\tau)^{\alpha-1}(\theta_1\|v-v^*\| + \nu_1\|D^\delta v - D^\delta v^*\|) d\tau$$
$$+\frac{t^{1-\sigma}}{\Gamma(2-\sigma)|\Delta|}\Big(\frac{\sum_{i=1}^k a_i}{\Gamma(\alpha+p)}\int_0^{b_i} (b_i-\tau)^{\alpha+p-1}(\theta_1\|v-v^*\|$$
$$+\nu_1\|D^\delta v - D^\delta v^*\|) d\tau + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}(\theta_1\|v-v^*\|$$
$$+\nu_1\|D^\delta v - D^\delta v^*\|) d\tau)$$

$$|D^\sigma T_1 v(t) - D^\sigma T_1 v^*(t)| \le \Big(\frac{2}{\Gamma(\alpha-\sigma+1)}$$

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$$+\frac{\sum_{i=1}^{k} a_i\ b_i^{\alpha+p}}{|\Delta|\Gamma(2-\sigma)\Gamma(\alpha+p+1)}(\theta_1+v_1)\times(\|v-v^*\| + \|D^{\delta}v-D^{\delta}v^*\|)$$

This implies that

$$\|D^{\sigma}T_1v-D^{\sigma}T_1v^*\| \leq D_2(\theta_1+v_1)(\|v-v^*\| + \|D^{\delta}v-D^{\delta}v^*\|)$$

Then we get

$$\|T_1v-T_1v^*\|_E \leq (D_1+D_2)(\theta_1+v_1)(\|v-v^*\| + \|D^{\delta}v-D^{\delta}v^*\| \quad (10)$$

With the same arguments as before, we have

$$\|T_1u-T_1v^*\|_E \leq (D_3+D_4)(\theta_2+v_2)(\|u-u^*\| + \|D^{\delta}u-D^{\delta}u^*\|) \quad \dots(11)$$

Applying the Banach contraction mapping principle, we conclude that T possesses a solitary fixed point that serves as the solution to equation (1). And From equations (10) and (11) then T is a contractive function. The Lemma yields the following result.

Lemma 3.1. Let X is Banach space and $S: X \longrightarrow X$ It operate continuouly and without interruption. then the set
$F = \{u \in X : u = \mu Su, 0 < \mu < 1\}$ S is bounded, and it has a fixed point within X.

Theorem 3.2. suppose that f, g: [u, v] × R × R are continuous function. In addition, we assume that:
(A1) asserts that the application of T is continuous when f and g are also continuous.

Proof.
Our approach will consist of four sequential steps:
Let's consider a set.

$B_r = \{(u,v) \in E \times F : \|(u,v)\|_{E\times F} \leq r\}, r > 0, then\ for\ t \in J,$ we have

$$|T_1v(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left|f(\tau,v(\tau),D^{\sigma}v(\tau))\right|d\tau$$

$$+\frac{t}{|\Delta|}\left(\frac{\sum_{i=1}^{k} a_i}{\Gamma(\alpha+p)}\int_0^{b_i}(b_i-\tau)^{\alpha+p-1}\left|f(\tau,v(\tau),D^{\sigma}v(\tau))\right|d\tau\right.$$

$$\left.+\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\left|g(\tau,v(\tau),D^{\sigma}v(\tau))\right|d\tau\right)$$

Thanks to (A3), we can write

$$|T_1v(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\sup_{t\in J} m_1(t)\,d\tau$$

$$+\frac{t}{|\Delta|}\left(\frac{\sum_{i=1}^{k} a_i}{\Gamma(\alpha+p)}\int_0^{b_i}(b_i-\tau)^{\alpha+p-1}\sup_{t\in J} m_1(t)\,d\tau\right)$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\sup_{t\in J} m_1(t)\,d\tau \leq \sup_{t\in J} m_1(t)\left[\frac{2}{\Gamma(\alpha+1)}+\frac{\sum_{i=1}^{k} a_i\,b_i^{\alpha+p}}{|\Delta|\Gamma(\alpha+p+1)}\right]$$

Therefore

$$\|T_1v\| \leq L_f D_1 \quad \dots(12)$$

On other hand we have,

$$|D^{\sigma}T_1v(t)| \leq \frac{1}{\Gamma(\alpha-\sigma)}\int_0^t (t-\tau)^{\alpha-\sigma-1}\left|f(\tau,v(\tau),D^{\sigma}v(\tau))\right|d\tau$$

$$+\frac{t^{1-\sigma}}{\Gamma(2-\sigma)|\Delta|}\left(\frac{\sum_{i=1}^{k} a_i}{\Gamma(\alpha+p)}\int_0^{b_i}(b_i-\tau)^{\alpha+p-1}\left|f(\tau,v(\tau),D^{\sigma}v(\tau))\right|d\tau\right.$$

$$\left.+\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\left|f(\tau,v(\tau),D^{\sigma}v(\tau))\right|d\tau\right)$$

And by (A3) we get

$$|D^{\sigma}T_1v(t)| \leq L_f\left(\frac{2}{\Gamma(\alpha-\sigma+1)}+\frac{\sum_{i=1}^{k} a_i\ b_i^{\alpha+p}}{|\Delta|\Gamma(2-\sigma)\Gamma(\alpha+p+1)}\right)$$

Consequently, we obtain,
$\|T_1v\| \leq L_f D_2 \quad , t \in J$
Then we have

$$\|D^{\sigma}T_1\| \leq L_f D_2 \quad \dots(13)$$

Thank to (12) and (13) we obtain

$$\|T_1v\|_E \leq L_f(D_1+D_2) \quad \dots(14)$$

The same method gives

$$\|T_1u\|_F \leq L_g(D_3+D_4) \quad \dots(15)$$

Thus using (14) and (15) we get

$$\|T(u,v)\|_{E\times F} \leq L_f(D_1+D_2)+L_g(D_3+D_4) \quad \dots(16)$$

Hence, for any (u, v) $\in B_r$; we obtain $\|T(u,v)\|_{E\times F} < +\infty$; which implies that the operator T is uniformly bounded on B_r then given sets into equal sets. $.E\times F$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and let Br be the above-bounded set E × F; ∀$(u, v) \in B_r$

$$T_1v(t_2)-T_1v(t_1) = \frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1}]f(\tau,v(\tau),D^{\sigma}v(\tau))d\tau$$

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$$+\frac{(t_2-t_1)}{\Delta}\left(\frac{\sum_{i=1}^{k} a_i}{\Gamma(\alpha+p)}\int_0^{b_i}(b_i-\tau)^{\alpha+p-1} f\big(\tau, v(\tau), D^\sigma v(\tau)\big)d\tau\right.$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f\big(\tau, v(\tau), D^\sigma v(\tau)\big)d\tau$$

From hypothesis (A3), we obtain:

$$|T_1v(t_2)-T_1v(t_1)| \le \frac{L_f}{\Gamma(\alpha)}\left|\int_0^{t_1}[(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}]d\tau\right| +$$

$$\frac{(t_2-t_1)}{\Delta}\left(\frac{\sum_{i=1}^{k} a_i}{\Gamma(\alpha+p)}\int_0^{b_i}(b_i-\tau)^{\alpha+p-1} f\big(\tau, v(\tau), D^\sigma v(\tau)\big)d\tau\right.$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f\big(\tau, v(\tau), D^\sigma v(\tau)\big)d\tau$$

Thus

$$|T_1v(t_2)-T_1v(t_1)| \le \frac{L_f}{\Gamma(\alpha)}\left|\int_0^{t_1}[(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}]d\tau\right| + \frac{L_f}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}d\tau\right| + \frac{L_f(t_2-t_1)}{|\Delta|}D_1 \quad \dots(17)$$

And

$$|D^\sigma T_1v(t_2)-D^\sigma T_1v(t_1)| \le \frac{L_f}{\Gamma(\alpha-\sigma)}\left|\int_0^{t_1}[(t_2-\tau)^{\alpha-\sigma-1}-(t_1-\tau)^{\alpha-\sigma-1}]d\tau\right| \quad \dots(18)$$

Now using (17) and (18) we have

$$\|T_1v(t_2)-T_1v(t_1)\|_E \le \frac{L_f}{\Gamma(\alpha)}\left|\int_0^{t_1}[(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}]d\tau\right| + \frac{L_f}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}d\tau\right| + \frac{L_f(t_2-t_1)}{|\Delta|}D_1$$
$$+\frac{L_f}{\Gamma(\alpha-\sigma)}\left|\int_0^{t_1}[(t_2-\tau)^{\alpha-\sigma-1}-(t_1-\tau)^{\alpha-\sigma-1}]d\tau\right| + \frac{L_f}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}d\tau\right| \quad \dots(19)$$

Using the same methods as before, of

$$\|T_1v(t_2)-T_1v(t_1)\|_F \quad \dots(20)$$

Then right-hand sides of equations (19) and (20) tend to zero apart from each other. The pair (u, v) transitions from time $t_1 \to t_2$. We can infer that T is completely continuous based on the Ascoli-Arzela theorem.

We will show that the set $S = \{(u,v) \in E \times F : (u,v) = \mu T(u,v); 0 < \mu < 1\}$ is bounded.

Let $(u,v) \in S$, then $(u,v) = \mu T(u,v)$ for $0 < \mu < 1$. For all $t \in J$ we have

Then we have

$$\frac{1}{\mu}|u(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left|f\big(\tau, v(\tau), D^\sigma v(\tau)\big)\right|d\tau + \frac{t}{|\Delta|}\left(\frac{\sum_{i=1}^{k} a_i}{\Gamma(\alpha+p)}\int_0^{b_i}(b_i-\tau)^{\alpha+p-1}\left|f\big(\tau, v(\tau), D^\sigma v(\tau)\big)\right|d\tau\right)$$
$$+\frac{t}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\left|f\big(\tau, v(\tau), D^\sigma v(\tau)\big)\right|d\tau$$

Using (A3) we have $\frac{1}{\mu}|u(t)| \le L_f D_1$ that is $|u(t)| \le \mu L_f D_1$

On the other hand, with a simple calculation we get

$$|D^\sigma u(t)| \le \mu L_f D_2$$

Consequently, we get

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$$\|u(t)\|_E \le \mu L_f(D_1 + D_2) \quad \dots(21)$$

The same computation gives

$$\|u(t)\|_F \le \mu L_f(D_3 + D_4) \quad \dots(22)$$

It follows from (21) and (22) that from (23), we see that

$$\| T(u,v) \|_{E\times F} \le +\infty. \quad \dots(23)$$

According to Schaefer's fixed-point theory, we observe that T possesses a fixed point, which serves as a solution to the dual system of the initial system.

4. Applications

In this section, we provide an example that shows our results.

$$\begin{cases} D^{\frac{5}{3}}u(t) = f\left(t, v(\tau), D^{\frac{1}{2}}v(\tau)\right), t \in J = [0,1] \\ D^{\frac{5}{3}}v(t) = f\left(t, v(\tau), D^{\frac{1}{3}}u(\tau)\right), t \in J = [0,1] \\ u(0) = 0, u(1) = 3I^{\frac{4}{3}}u\left(\frac{6}{5}\right) + I^{\frac{4}{3}}u\left(\frac{5}{2}\right) + \frac{7}{5}I^{\frac{4}{3}}u\left(\frac{7}{6}\right) \\ v(0) = 0, v(1) = \frac{4}{3}I^{\frac{5}{4}}u\left(\frac{5}{2}\right) + I^{\frac{5}{4}}u\left(\frac{3}{2}\right) + \frac{7}{4}I^{\frac{5}{4}}u\left(\frac{6}{5}\right) \end{cases}.$$

$$f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$$

$$(t,u,v) \to \frac{2}{e^{t+5}(1 + |u| + |v|)} + t \sin t$$

$$g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$$

$$(t,u,v) \to \frac{2}{e^{t+7}(1 + |u| + |v|)} + t \sin t$$

$$\left|\frac{2}{e^{t+5}(1 + |u| + |v|)} - \frac{2}{e^{t+5}(1 + |\bar{u}| + |\bar{v}|)}\right| \le \frac{2}{e^{t+5}}(|u - \bar{u}| + |v - \bar{v}|)$$

and

$$\left|\frac{2}{e^{t+7}(1 + |u| + |v|)} - \frac{2}{e^{t+7}(1 + |\bar{u}| + |\bar{v}|)}\right| \le \frac{2}{e^{t+7}}(|u - \bar{u}| + |v - \bar{v}|)$$

So, we can take

$$\lambda_1(t) = \nu_1(t) = \frac{2}{e^{t+5}}$$

and

$$\lambda_2(t) = \nu_2(t) = \frac{2}{e^{t+7}}$$

It follows that

$$\theta_1 = \nu_1 = \frac{2}{e^5} \ and\ \theta_2 = \nu_2 = \frac{2}{e^7}$$

that give

$|f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w})| \le c_1|u - \bar{u}| + c_2|v - \bar{v}| + c_3|w - \bar{w}|$where $c_1 = c_2 = c_3 = 10\, e^5$.

Hence, hypotheses (H1) and (H2) are satisfied.
We can get that $D < 1$ with a simple calculation. we get. $D_1 = 0{,}0807$ and $D_1 = 0{,}24$ and .
That implies... $D < 1$ Thus, all the assumptions from (A1)–(A3) are satisfied. From theorem 3, we conclude that equation (1) has a unique solution.
on (1) has a unique solution.

5. Conclusions

This research focuses on investigating the presence and singularity of solutions to the fractional Caputo differential equation system, considering initial boundary conditions. Banach's fixed point theorem can demonstrate the existence of solutions to Caputo's system. There is no text provided. The theorem was utilized to establish the existence, while the singularity conclusion was derived using Schaefer's fixed-point theorem. Lastly, we present an example to demonstrate our findings.

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