A New Framework of bi-capacities and its Integrals

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Abstract

The aim of this paper is to introduce a new framework for studying bi-capacities and its integrals, which is alternative framework from defined by Grabisch and Labreuche ([7], [8]). First, we define bi-capacities through introducing a notion of ternary-element sets. Secondly, we propose model of bipolar Choquet integral with respect to bi-capacity based on ternary-element sets, and we study basic properties of new model.

Keywords: Capacities, Bi-cpacities, Choquet integral, Bipolar Choquet integral, Ternary-element sets.

1. Introduction

The area of applications of capacity [3] (also known under the name of fuzzy measure in some literatures [6], [10]) and Choquet integral (see, e.g., [3], [5], [1], [6]) has been greatly expanded in the different fields, especially in the field of decision theory. However, capacity and Choquet integral are not suitable in some situations such as decision behaviors, in particular when the underlying scales are bipolar. In [7], the notion of bi-capacity was proposed (as a natural extension of capacity) by Grabisch and Labreuche to resolve these problems. The Choquet integral with respect to bi-capacities has been introduced by Grabisch and Labreuche [8] as a generalization of the Choquet integral.

The paper aims at constructing a different framework from defined by Grabisch and Labreuche ([7], [8]) suited to bipolar scales. First, we present a new framework for studying capacities and its integrals through introducing a notion of binary-element sets. Then, we extend the notion of binary-element sets to ternary-element sets, and propose a new framework of bi-capacities and bipolar Choquet integral with respect to bi-capacity based on a notion of ternary-element sets.

The structure of the paper is as follows. In the next section we recall basic definitions of usual capacities, bi-capacities and its integrals. The third section presents a new framework of capacities and its integrals. In section 4, we propose a new framework of bi-capacities and bipolar Choquet integral with respect to bi-capacities. In section 5, we study basic properties of new framework. The paper finishes with some conclusions.

2. Usual capacities, bi-capacities and its integrals

In this section we give some basic definitions of usual capacities, bi-capacities and its integrals. Usual capacities on some finite universe X are special monotone set functions defined in the following way (see, e.g. [3], [5]).

Definition 1:

A capacity on X is a set function $g: 2^X \to [0,1]$ satisfying the following requirements (i) $g(\emptyset) = 0 \ and \ g(X) = 1$,

(ii) $\forall A, B \in 2^X, A \subseteq B \ implies$

$$g(A) \leq g(B).$$

In a unbipolar context, when using a capacity to model the importance of the subsets of criteria, a suitable aggregation operator that generalizes the weighted arithmetic mean is the Choquet integral (see, e.g. [5], [6]) which is defined as follows.

Definition 2:

Let g be a capacity on X. The Choquet integral of a measurable function $h: X \to [0,1]$ with respect to g, which is denoted by $(C)\int h\, dg$, can be defined as

$$(C)\int h\, dg = \sum_{i^*=1}^{n^*} [h(i^*) - h(i^*-1)]\, g(\{i^*, \cdots, n^*\}) \quad \text{.................................. (1)}$$

where $(1^*, 2^* \cdots, n^*)$ is a permutation of $\{1,2 \cdots, n\}$ so that $h(1^*) \leq \cdots \leq h(n^*)$ with a convention $h(0) = 0$.

An alternative equivalent expression for equation (1) is

$$(C)\int h\,dg = \sum_{i^*=1}^{n^*} h(i^*)\,[g(\{i^*,\cdots,n^*\}) - g(\{i^*+1,\cdots,n^*\})] \quad \text{.......} (2)$$

with a convention $g(\{n^*+1\}) = 0$.

As a natural extension of capacities when the underlying evaluation scale is bipolar, Grabisch and Labreuche [7] have proposed bi-capacities.

Let $3^X := \{(A,B) \in 2^X \times 2^X | A \cap B = \emptyset\}$ be the set of all ordered pairs of disjoint coalitions, and endow it with a partial order $\subseteq$ defined by $(A,B) \subseteq (A',B')$ iff $A \subseteq A'$ and $B \supseteq B'$.

***Definition 3*:** (Bi-capacity)

A function $\mathcal{V}: 3^X \to [-1,1]$ is a bi-capacity on 3^X if it satisfies:

(i) $\mathcal{V}(\emptyset,\emptyset) = 0, \mathcal{V}(X,\emptyset) = 1, \mathcal{V}(\emptyset,X) = -1$,
(ii) $(A,B),(A',B') \in 3^X, (A,B) \subseteq (A',B')$ $implies\ \mathcal{V}(A,B) \le \mathcal{V}(A',B')$.

In a bipolar context, when bi-capacities are used to model the importance of the subsets of criteria, Grabisch and Labreuche [8] proposed the following natural extension of the Choquet integral.

Definition 4: (Choquet integral w.r.t. bi-capacities)

Let $\mathcal{V}$ be a bi-capacity and f be a real-valued function on X. The Choquet integral of f w.r.t $\mathcal{V}$ is given by

$$\int f\, d\mathcal{V} = \int |f|\, d\mathcal{V}_{X_f^+}$$

Where $\mathcal{V}_{X_f^+}$ is a set function on X vanishing at the empty set defined by

$$v_{X_f^+}(C) := v(C \cap X_f^+, C \cap X_f^-)$$

and $X_f^+ := \{i \in X | f_i \ge 0\}, X_f^- := X \backslash X_f^+$.

3. A new approach of capacities and its integrals

3.1 Capacities based on bi-element sets

In the context of multi-criteria decision making problem, we shall consider a set $X = \{1,2\cdots,n\}$ of criteria and a set of alternatives described according to these criteria. Since the criteria have not always the same importance, and defining a capacity on a set of criteria can be seen as a way of modeling the interaction phenomena existing among these criteria. Hence, for every element $i \in X$ has either positive effect (i.e., i is positively important criterion of weighted evaluation not only alone but also is interactive with other) or has negative effect (i.e., i is negatively important criterion). Thus, we can represent the element i as i^+ whenever i is positively important, and as i^- whenever i is negatively important, and we call this element a binary-element (or simply bi-element). The binary-element set (or simply bi-element set) is the set which contains either i^+ or i^- for all $i, i = 1,2\cdots,n$. For this situation, we can describe the set of all possible combinations of binary elements of n criteria given by $\mathcal{B}(X) = \{\ \{\tau_1,\cdots,\tau_n\} | \ \forall\, \tau_i \in \{i^+, i^-\},\ i = 1,2\cdots,n\}$ which is corresponds to power set $\mathcal{P}(X)$ in the notation of classical set theory.

The set of all bi-element sets $\mathcal{B}(X)$ can be identified with $\{0,1\}^2$, hence

$|\mathcal{B}(X)| = 2^n$. Also, any bi-element set $A \in \mathcal{B}(X)$ can be written as a binary alternative denoted by $(\tau_1,(\tau_2,\cdots,\tau_n)$ with $\tau_i = 1\ if$ $i^+ \in A$ and $\tau_i = 0$ if $i^- \in A$, $i = 1,2\cdots,n$.

We introduce the inclusion relation $\subseteq$ of bi-element sets of $\mathcal{B}(X)$as follows.

Definition 5 :

Let $\mathcal{B}(X)$ be the set of all bi-element sets and $A,B \in \mathcal{B}(X)$.

Then, $A \subseteq B\ iff\ \ \forall\ i \in X$,
$if\ \ i^+ \in A\ implies\ i^+ \in B$.

Note that, $X^- = \{1^-,2^-\cdots,n^-\} \subseteq A$ and $X^+ = \{1^+,2^+\cdots,n^+\} \supseteq A$, $\forall\, A \in \mathcal{B}(X)$.

Based on the notion of bi-element sets, the following definition gives an equivalent definition of capacities.

Definition 6:

Let $\mathcal{B}(X)$ be the set of all bi-element sets. A set function

$\mu: \mathcal{B}(X) \to [0,1]$ is called capacity if it satisfies the following requirements:

(i) $\mu(X^-) = \mu(\{1^-,2^-\cdots,n^-\}) = 0$ and $\mu(X^+) = \mu(\{1^+,2^+\cdots,n^+\}) = 1$.
(ii) $\forall\, A,B \in \mathcal{B}(X),\ A \subseteq B\ implies\ \mu(A) \le \mu(B)$.

3.2 The Choquet integral based on the bi-element sets

In this subsection, we propose an equivalent model of Choquet integral with respect to capacities based on the bi-element sets. We assume that to each alternative in multi-criteria decision making problem is described by a vector $x = (x_{1^+}, \cdots, x_{i^+}, \cdots, x_{n^+})$; $x_{i^+} \in R$ with $i \in \{1,2 \cdots, n\}$, and we consider a bi-element set $X^+ = \{i^+ | x_{i^+} \in R$ for each $i \in \{1,2 \cdots, n\}\}$. Thus, we define the Choquet integral with respect to capacity of real input $\boldsymbol{x}$ as follows.

Definition 7:

Let $\mathcal{B}(X)$ be the set of all bi-element sets and $\mu: \mathcal{B}(X) \to [0,1]$ be a capacity based on bi-element set. Then Choquet integral of $\boldsymbol{x}$ with respect to μ is given by

$$Ch_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} [x_{\sigma(i^+)} - x_{\sigma((i-1)^+)}] \; \mu\left(\left\{A_{\sigma(i^+)}\right\}\right) \ldots (3)$$

where $A_{\sigma(i^+)} = \{\ldots, \sigma((i-2)^-), \sigma((i-1)^-), \sigma(i^+), \ldots, \sigma(n^+)\}$ is bi-element set $\subseteq X^+$, and σ is a permutation on X^+ so that $x_{\sigma(i^+)} \le \cdots \le x_{\sigma(n^+)}$ with the convention $x_{\sigma((0)^+)} \coloneqq 0$.

An equivalent expression for the equation (3) is

$$Ch_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} x_{\sigma(i^+)} \; [\mu(\{A_{\sigma(i^+)}\}) - \mu(\{A_{\sigma((i+1)^+)}\})] \ldots\ldots (4)$$

with the same notation above and $\mu(\{A_{\sigma((n+1)^+)}\}) \coloneqq 0$.

From the definition of the Choquet integral with respect to capacity based on bi-element set, we immediately see the following property.

Proposition 1:

Let $\mathcal{B}(X)$ be the set of all bi-element sets and $\mu: \mathcal{B}(X) \to [0,1]$ be a capacity based on bi-element set. Then Choquet integral of $\boldsymbol{x}$ with respect to μ is given by

$$Ch_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} [x_{\sigma(i^+)} - x_{\sigma((i+1)^+)}] \; \mu\left(\{A_{\sigma(i^+)}\}\right) \ldots\ldots (5)$$

or as

$$Ch_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} x_{\sigma(i^+)} \; [\mu\left(\{A_{\sigma(i^+)}\}\right) - \mu(\{A_{\sigma((i-1)^+)}\})] \ldots\ldots (6)$$

where

$A_{\sigma(i^+)} = \{\sigma(1^+), \ldots, \sigma(i^+), \sigma((i+1)^-), \sigma((i+2)^-), \ldots\}$ is bi-element set $\subseteq X^+$, and σ is a permutation on X^+ so that $x_{\sigma(i^+)} \ge \cdots \ge x_{\sigma(n^+)}$ with the convention $x_{\sigma((n+1)^+)} \coloneqq 0$. and $\mu(\{A_0\}) \coloneqq 0$.

For the sake of clarity, let us give the following numerical example.

Example 1:

For $n = 3$, let us consider $x = (4,6,-3)$. Applying the Choquet integral with respect to the capacity based on bi-element sets (Formula (5)) we obtain

$Ch_\mu(4,6,-3) = (6-4)\,\mu(\{2^+,1^-,3^-\}) + (4-(-3))\,\mu(\{2^+,1^+,3^-\}) + (-3-0)\,\mu(\{2^+,1^+,3^+\}) = 2\mu(\{2^+,1^-,3^-\}) + 7\,\mu(\{2^+,1^+,3^-\}) - 3\mu(\{2^+,1^+,3^+\})$.

4. A new approach of bi-capacities and its integrals

4.1 Bi-capacities based on ter-element sets

In this subsection, we will extend the scale of capacity to a bipolar scale [-1, 1] and we generalize the concept of binary-element sets to ternary-element sets, then we give equivalent definitions of bi-capacities based on ternary-element sets. So we are going to work on the scale [-1, 1] and we will say that [0, 1] is the degree of satisfaction (or "positive part") and [-1, 0] the degree of non satisfaction (or "negative part"), where 1 is the full satisfaction and -1 the full non satisfaction. In this construction an interesting point is 0. It is the middle point between the full satisfaction and the full non-satisfaction. So we will consider that it represents the "neutral part".

The idea is to extend the notion of bi-element sets, and define a concept that gathers all combinations of positive, negative, and neutral values on the criteria. For this situation, we assume that, for every element $i \in X$ has either positive effect (i.e., i is positively important criterion), or has negative effect (i.e., i is negatively important criterion), or has no effect (i.e., i is criterion at neutral level). Hence, we represent the element i as i^+ whenever i is positively important, as i^- whenever i is negatively important, and as $i^\emptyset$ whenever i is neutral, and we call this element a ternary-element (or simply ter-element). The ternary-element set (or simply ter-element set) is the set which contains only out of $i^+, i^-, and\ i^\emptyset$ for all $i, i \in \{1,2 \cdots, n\}$.

Thus, in our model we consider the set of all possible combinations of ternary elements of n criteria given by

$$\mathcal{T}(X) := \{\{\tau_1, \cdots, \tau_n\} | \forall\ \tau_i \in \{i^+, i^-, i^\emptyset\}, \quad i = 1,2 \cdots, n\}$$

which is corresponds to $Q(X)$ in the notation of classical bi-capacities ([7]).

Note that, $\mathcal{T}(X)$ can be identified with $\{-1, 0,1\}^n$, hence $|\ \mathcal{T}(X)| = 3^n$.

Also, simply remark that for any ter-element set $A \in \mathcal{T}(X), A$ is equivalent to a ternary alternative $(\tau_1, \cdots, \tau_n)$ with

$\tau_i = 1$ if $i^+ \in A$, $\tau_i = 0$ if $i^\emptyset \in A$, and $\tau_i = -1$ if $i^- \in A, \forall\ i = 1,2 \cdots, n$.

We introduce the order relation $\sqsubseteq$ between ter-element sets of $\tau\ (X)$ as follows.

Definition 8 :

Let $\mathcal{T}\ (X)$ be the set of all ter-element sets and $A, B \in\ \mathcal{T}\ (X)$.

Then, $A\ \sqsubseteq\ B\ iff\ \forall\ i\ \in\ X$, "$if\ i^+\ \in\ A\ implies\ \ i^+\ \in\ B$", and "$if\ i^\emptyset\ \in\ A\ implies\ i^+\ or\ i^\emptyset\ \in B$" .... (7)

Note that, $X^- = \{1^-, 2^- \cdots, n^-\} \sqsubseteq A$ and $X^+ = \{1^+, 2^+ \cdots, n^+\} \sqsupseteq A$, $\forall\ A \in\ \mathcal{T}(X)$.

The following definition is equivalent definition of bi-capacities based on notion of ter-element sets.

Definition 9 :

Let $\mathcal{T}\ (X)$ be the set of all ter-element sets. A set function $v : \mathcal{T}\ (X) \to\ [-1, 1]$, is called bi-capacity if it satisfies the following requirements:

(i) $v(X^-) = v(\{1^-, 2^- \cdots, n^-\}) = -1$, $v(X^\emptyset) = v(\{1^\emptyset, 2^\emptyset \cdots, n^\emptyset\}) = 0$, and $v(X^+) = v(\{1^+, 2^+ \cdots, n^+\}) = 1$.

(ii) $\forall\ A, B \in\ \tau(X)$, $\quad A \sqsubseteq B\ implies\ v(A) \leq v(B)$.

Bi-capacity based on ter-element set $v(A), \forall\ A\ \in\ \mathcal{T}\ (X)$is exactly the overall score or utility of the ternary alternative $(\tau_1, \cdots, \tau_n)$ with $\tau_i = 1$ (full satisfaction) if $i^+ \in A$, i.e. on positively important elements in $A, \tau_i = -1$ (full non satisfaction) if $i^- \in A$, i.e. on negatively important elements in $A, and\ \tau_i = 0$ (neutral) if $i^\emptyset \in A$, i.e. according to the neutral elements in A.

4.2 The order $\subseteq$ on $\mathcal{T}\ (X)$

Bi-capacities are functions defined on the structure of the underlying partially ordered set [4]. There are several orders on the structure $Q(X)$ have introduced by Grabisch and Labreuche [7, 9] and Bilbao et al. [2]. In this subsection, we introduce an order on the structure $\mathcal{T}\ (X)$ different from the order $\sqsubseteq$ described in subsection 4.1 (Definition 8).

We consider the following definition of an order on $\mathcal{T}\ (X)$ which is equivalent to Bilbao order on bi-cooperative game [2]. For convenience, we denote by $\subseteq$ the order relation defined on $\mathcal{T}\ (X)$ as in the definition 5, and we will use the order $\subseteq$ on $\mathcal{T}\ (X)$ to establish our next results of this research.

Definition 10 :

Let $\mathcal{T}\ (X)$ be the set of all ter-element sets and $A, B\ \in\ \mathcal{T}\ (X)$. Then, $A \subseteq\ B\ iff\ \ \forall\ i\ \in X$, "$if\ i^+\ \in\ A\ implies\ i^+\ \in B$", and "$if\ i^-\ \in\ A\ implies\ i^-\ \in\ B$". .... (8)

4.3 The bipolar Choquet integral based on the ter-element Sets

In this subsection, we propose Choquet integral model with respect to bi-capacity based on the ter-element set of real input $\boldsymbol{x}$. The basic idea underlying this model is for an input vector $\boldsymbol{x} = (x_{\tau_1}, \ldots, x_{\tau_i}, \ldots, x_{\tau_n})$; $x_{\tau_i} \in R\ with\ i \in \{1,2 \cdots, n\}$. we consider a ter-element set $X^* = \{\tau_1, \cdots, \tau_n\}$ with $\tau_i = i^+\ if\ x_i > 0;\ \tau_i = i^-\ if\ x_i < 0;\ and\ \tau_i = i^\emptyset\ if\ x_i = 0, \forall\ i = 1, \ldots, n$.

Thus, we define the bipolar Choquet integral with respect to bi-capacity based on ter-element set of real input $\boldsymbol{x}$ as follows.

Definition 11 :

Let $\mathcal{T}\ (X)$ be the set of all ter-element sets and $v :\ \mathcal{T}\ (X) \to [-1, 1]$, be a bi-capacity based on ter-element set.

Then, the bipolar Choquet integral of $\boldsymbol{x}$ with respect to v is given by

$$Ch_v(\boldsymbol{x}) = \sum_{i=1}^{n} [\ |x_{\sigma(\tau_i)}| - |x_{\sigma(\tau_{(i-1)})}|]\ v(A_{\sigma(\tau_i)}),$$

$\tau_i \in \{i^+, i^-, i^\emptyset\}$ .... (9)

where $A_{\sigma(\tau_i)} = \{\dots, \sigma((i-2)^{\emptyset}), \sigma((i-1)^{\emptyset}), \sigma(\tau_i), \dots, \sigma(\tau_n)\}$ is ter-element set $\subseteq X^*$, and σ is a permutation on X^* so that $|x_{\sigma(\tau_i)}| \leq \cdots \leq |x_{\sigma(\tau_n)}|$ with the convention $x_{\sigma(0)} := 0$.

An equivalent expression for the equation (9) is

$$Ch_v(\boldsymbol{x}) = \sum_{i=1}^{n} |\, x_{\sigma(\tau_i)} \,| [\, v(\{A_{\sigma(\tau_i)}\}) - v(\{A_{\sigma(\tau_{i+1})}\})] \dots \quad \text{(10)}$$

with the same notation above and $v(\{A_{\sigma(n+1)}\}) := 0$.

From the definition of the bipolar Choquet integral with respect to the bi-capacity based on ter-element set, we immediately see the following property.

Proposition 2 :

Let $\mathcal{T}(X)$ be the set of all ter-element sets and $v : \mathcal{T}(X) \to [-1, 1]$, be a bi-capacity based on ter-element set. Then, the bipolar Choquet integral of $\boldsymbol{x}$ with respect to v is given by

$$Ch_v(\boldsymbol{x}) = \sum_{i=1}^{n} [\, |x_{\sigma(\tau_i)}| - |x_{\sigma(\tau_{(i+1)})}|]\; v(\{A_{\sigma(\tau_i)}\}), \quad \tau_i \in \{i^+, i^-, i^{\emptyset}\} \quad \text{(11)}$$

Or as

$$Ch_v(\boldsymbol{x}) = \sum_{i=1}^{n} |\, x_{\sigma(\tau_i)} \,| \; [\, v(\{A_{\sigma(\tau_i)}\}) - v(\{A_{\sigma(\tau_{i-1})}\})] \quad \text{(12)}$$

where $A_{\sigma(\tau_i)} = \{\sigma(\tau_1), \dots, \sigma(\tau_i), \sigma((i+2)^{\emptyset}), \sigma((i+1)^{\emptyset}), \dots\}$ is ter-element set $\subseteq X^*$, and σ is a permutation on X^* so that $|x_{\sigma(\tau_i)}| \geq \cdots \geq |x_{\sigma(\tau_n)}|$ with the convention $x_{\sigma(\tau_{n+1})} := 0 \; and \; v(A_{\sigma(0)}) := 0$.

The following numerical example illustrates the use of the bipolar Choquet integral with respect to bi-capacity based on the ter-element sets.

Example 2:

[Example 1 continued] Applying the bipolar Choquet integral with respect to bi-capacity based on the ter-element sets (Formula (11)) we obtain

$$Ch_v(4,6,-3) = (6-4)v(\{2^+, 1^{\emptyset}, 3^{\emptyset}\}) + (4-3)\, v(\{2^+, 1^+, 3^{\emptyset}\}) + (3-0)v(\{2^+, 1^+, 3^-\})$$
$$= 2v(\{2^+, 1^{\emptyset}, 3^{\emptyset}\}) + v(\{2^+, 1^+, 3^{\emptyset}\}) + 3v(\{2^+, 1^+, 3^-\}).$$

5. Properties of bipolar Choquet integral based on the ter-element sets

In this subsection, we give some basic properties satisfied by the proposed framework of bipolar Choquet integral with respect to the bi-capacity based on the ter-element sets.

Proposition 3 :

For positive input vectors the bipolar Choquet integral with respect to bi-capacity and Choquet integral with respect to capacity coincide.

***Proof*:**

Clear from the definition of the bipolar Choquet integral with respect to the bi-capacity based on the ter-element sets. ■

The next property shows the bipolar Choquet integral of ternary alter-natives $(1_A, -1_A, 0_A)$ equals bi-capacity $v(A), \forall A \in \mathcal{T}(X)$ as shown by the following result.

Proposition 4 :

For any bi-capacity based on the ter-element sets (v) on

$$\mathcal{T}(X), \quad Ch_v(1_A, -1_A, 0_A) = v(A), \quad \forall A \in \mathcal{T}(X).$$

***Proof*:**

For ternary vector $(1_A, -1_A, 0_A)$, $|x_{\sigma(\tau_i)}| = 1 \; or \; |x_{\sigma(\tau_i)}| = 0, \forall \tau_i \in \{i^+, i^-, i^{\emptyset}\}$ and $v(\{A_{\sigma(\tau_i)}\}) = v(1_A, -1_A, 0_A) = v(A)$.

Therefore, from the bipolar Choquet integral with respect to bi-capacity based on the ter-element sets (Formula (11)), we have

$$Ch_v(1_A, -1_A, 0_A) = \sum_{i=1}^{n} [\, |x_{\sigma(\tau_i)}| - |x_{\sigma(\tau_{(i+1)})}|]\; v(\{A_{\sigma(\tau_i)}\})$$
$$Ch_v(1_A, -1_A, 0_A) = v(A), \quad \forall A \in \mathcal{T}(X) \; ■$$

The next property shows that the bipolar Choquet integral with respect to the bi-capacity satisfy the monotonicity.

Proposition 5: (Monotonicity)

For any bi-capacity based on the ter-element sets v on $\mathcal{T}(X), \forall \boldsymbol{x}, \boldsymbol{x'} \in R$, $if\ x_{\tau_i} \geq x'_{\tau_i}\ \forall\ \tau_i \in \{i^+, i^-, i^\emptyset\}$, then $Ch_v(\boldsymbol{x}) \geq Ch_v(\boldsymbol{x'})$.

***Proof*:** First, we assume that for any $\tau_i \in \{i^+, i^-, i^\emptyset\}$, $x_{\tau_i} > x'_{\tau_i}$ and $\forall\ k \in \{1, \ldots, i-1, i+1 \ldots, n\}$, $x_{\tau_k} = x'_{\tau_k}$.

Also, we assume that for all elements of X, the order of each element is the same, i.e., $|x_{\sigma(\tau_i)}| \geq \cdots \geq |x_{\sigma(\tau_n)}|$ and $|x'_{\sigma(\tau_i)}| \geq \cdots \geq |x'_{\sigma(\tau_n)}|$.

Firstly, we prove the monotonicity of this case.

Using the equivalent expression of bipolar Choquet integral with respect to bi-capacity based on the ter-element sets (Formula (12)), we have

$$Ch_v(\boldsymbol{x}) = \sum_{i=1}^{n} |\, x_{\sigma(\tau_i)}\,| \ \left[v\left(\{A_{\sigma(\tau_i)}\}\right) - v(\{A_{\sigma(\tau_{i-1})}\ \}) \right] \quad \ldots\ldots (13)$$

also,

$$Ch_v(\boldsymbol{x'}) = \sum_{i=1}^{n} |\, x'_{\sigma(\tau_i)}\,| \ \left[v\left(\{A_{\sigma(\tau_i)}\}\right) - v(\{A_{\sigma(\tau_{i-1})}\ \}) \right] \quad \ldots\ldots (14)$$

$A_{\sigma(\tau_i)}\ and\ A_{\sigma(\tau_{i-1})}$ are the ternary-element sets with $A_{\sigma(\tau_{i-1})} \subseteq A_{\sigma(\tau_i)}$.

Hence, $v\left(\{A_{\sigma(\tau_i)}\}\right) - v(\{A_{\sigma(\tau_{i-1})}\ \}) \geq 0$.

Now, Since $x_{\tau_i} \geq x'_{\tau_i}\ \forall\ \tau_i \in \{i^+, i^-, i^\emptyset\}$, it is clear that $Ch_v(\boldsymbol{x}) \geq Ch_v(\boldsymbol{x'})$.

Therefore, if $x_{\tau_i} > x'_{\tau_i}$ then $Ch_v(\boldsymbol{x}) \geq Ch_v(\boldsymbol{x'})$ is proved within the range that the order of values of each element of $\boldsymbol{x}$ and $\boldsymbol{x'}$ does not change. Thus, by repeating the above procedures two times at the point of the change of the order, if $x_{\tau_i} > x'_{\tau_i}$ then $Ch_v(\boldsymbol{x}) \geq Ch_v(\boldsymbol{x'})$ can be proved even in the range with the change of the order. By applying this procedure nsuccessively for each i, the proposition can be proved. ■

6 Conclusions

In this paper, we have proposed an alternative expressions of bi-capacities and Choquet integrals with respect to bi-capacities. We have defined bi-capacities and bipolar Choquet integrals with respect to bi-capacities based on notion of ternary-element sets. Definition of bi-capacities and its integrals via notion of ter-element sets is important because attaching a polarity to each element is an easier than attaching a polarity to sets. Thus, we expect these definitions allow a simple way to define other integrals on bi-capacities.

References

[1] Abbas J., "Logical Twofold Integral", Engineering & Techology Journal, Vol. 28, No. 3, 2010.

[2] Bilbao J. M., Fernandez, J.R., Jimenez Losada, A. and Lebron, E., "Bicooperative Games", First World Congress of the Game Theory Society (Games 2000) July 24-28, Bilbao, Spain, 2000.

[3] Choquet G., "Theory of capacities", Ann. Inst. Fourier, Vol. 5, pp.131-295, 1953.

[4] Davey B.A., Priestley H.A., "Introduction to Lattices and Orders", Cambridge University Press, Cambridge, 1990.

[5] Denneberg D., "Non-additive measure and integral", Kluwer Academic Publisher, 1994.

[6] Grabisch, M., Murofushi T. and Sugeno M., "Fuzzy Measures and in tegrals", Physic-Verlag, 2000.

[7] Grabisch, M., Labreuche C., "Bi-capacities I: Definition, Mobius transform and interaction", Fuzzy Sets and Systems 151, pp. 211-236, 2005.

[8] Grabisch M., Labreuche C., "Bi-capacities II: The Choquet integral", Fuzzy Sets and Systems 151, pp. 237-259, 2005.

[9] Grabisch M. and Labreuche C., "A general construction for unipolar and bipolar interpolative aggregation", in: Joint Fourth EUSFLAT and 11th LFA Conference (EUSFLAT 2005), Barcelona, Spain, pp. 916921, 2005.

[10] Sugeno M., "Theory of fuzzy integrals and its applications", Ph.D. Thesis, Tokyo Institute of Technology, 1974.

الخلاصة

الهدف من هذا البحث هو تقديم اطار جديد لدراسة السعات ذات القطبية الثنائية وتكاملاتها، فهو اطار بديل عن المعرف في [٧] و [٨]. نَعرفَ اولا سعات ذات القطبية الثنائية من خلال تقديم فكرة مجموعات العنصر الثلاثي. ثانيا نَقترح موديل لتكامل جوكيت ذو القطبية الثنائية متعلق با السعات ذات القطبية الثنائية المستندة على مجموعات العنصر الثلاثي، و ندرس الخواص الأساسية للموديل الجديد.