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Some Methods to Estimate the Parameters of IDAL-Distribution (b)

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Abstract

This paper estimates three parameters for a new model called "IDAL Distribution". This model expands the exponential Weibull distribution by adding a third parameter and studying its characteristics. Several estimation methods, such as maximum likelihood estimation, ordinary least squares estimation, and proposed ordinary least squares methods, have been used to investigate the unknown parameters of the IDAL distribution. The goodness of fit of the proposed distribution is ultimately compared, and the results of the applied approaches are evaluated based on MSE.

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1. Introduction

Recently, the scientific and mathematical efforts of developing a new probability distribution have been carried out to take advantage of these distributions in several mathematical applications and different life fields. A new family of probability distributions has been introduced in several kinds of literature by adding new parameters to the basic and original distribution [1-4]. Some approaches to crating the new distribution are based on taking the benefit of the classical and basic distributions, such as exponential, Rayleigh, Weibull, or any other distributions of lifetime distributions and adding other parameters in order to increase the flexibility of the suggested distribution [5],[6]. A proposed method for calculating the reliability function of the stress-strength models of the power Lomax distribution has been carried out using various estimation methods in [7]. Two new distributions as special cases of the Exponentiated Lomax Distribution, forming a modified and restricted exponentiated Lomax distribution, were introduced in [8]. Several researchers have expanded the extent of the Weibull distribution and introduced a mixture of distributions in a newer area of study that has gained significant traction in statistical research articles, particularly in reliability analysis applications [9-13]. The distribution elucidates the methodology for combining Weibull distributions using a mixing Weibull distribution and adding two parameters that denote the proportions of the amalgamation of the two components of Weibull distributions that have been introduced in [14]. Other researchers presented a model of combining Poisson and Weibull distributions to analyze consumer behavior and its characteristics [15]. A new distribution by forming a log-logistic distribution with a Weibull distribution as a mechanism for generating composition distribution has been discussed in [16]. The updated model used the ratio of two separate random variables as a novel lifespan model that has been presented in [17]. Two new distributions were introduced as a special case of the Exponentiated Lomax Distribution, forming a family for the Exponential Lomax Distribution that was included in [18]. In this article a new lifetime distribution called the IDAL distribution with three parameters has been proposed to provide greater flexibility and efficiency in utilizing this distribution in different area of statistical studies and various mathematical fields, such as reliability analysis. The proposed methodology involved the estimation and simulation process of the three parameters of our new distribution using three methods: MLE, OLS, and proposed OLS, and a comparison between the obtained results via the MSE criterion to realize the best method and investigate the performance of the proposed distribution.

2. Genesis of IDAL Distribution

We constructed our new distribution, the IDAL Distribution, by adding the shape parameter to the Exponential Weibull Distribution and studied its properties in research [18]. It is possible to describe the proposed IDAL distribution family as depicted in Figure 1.

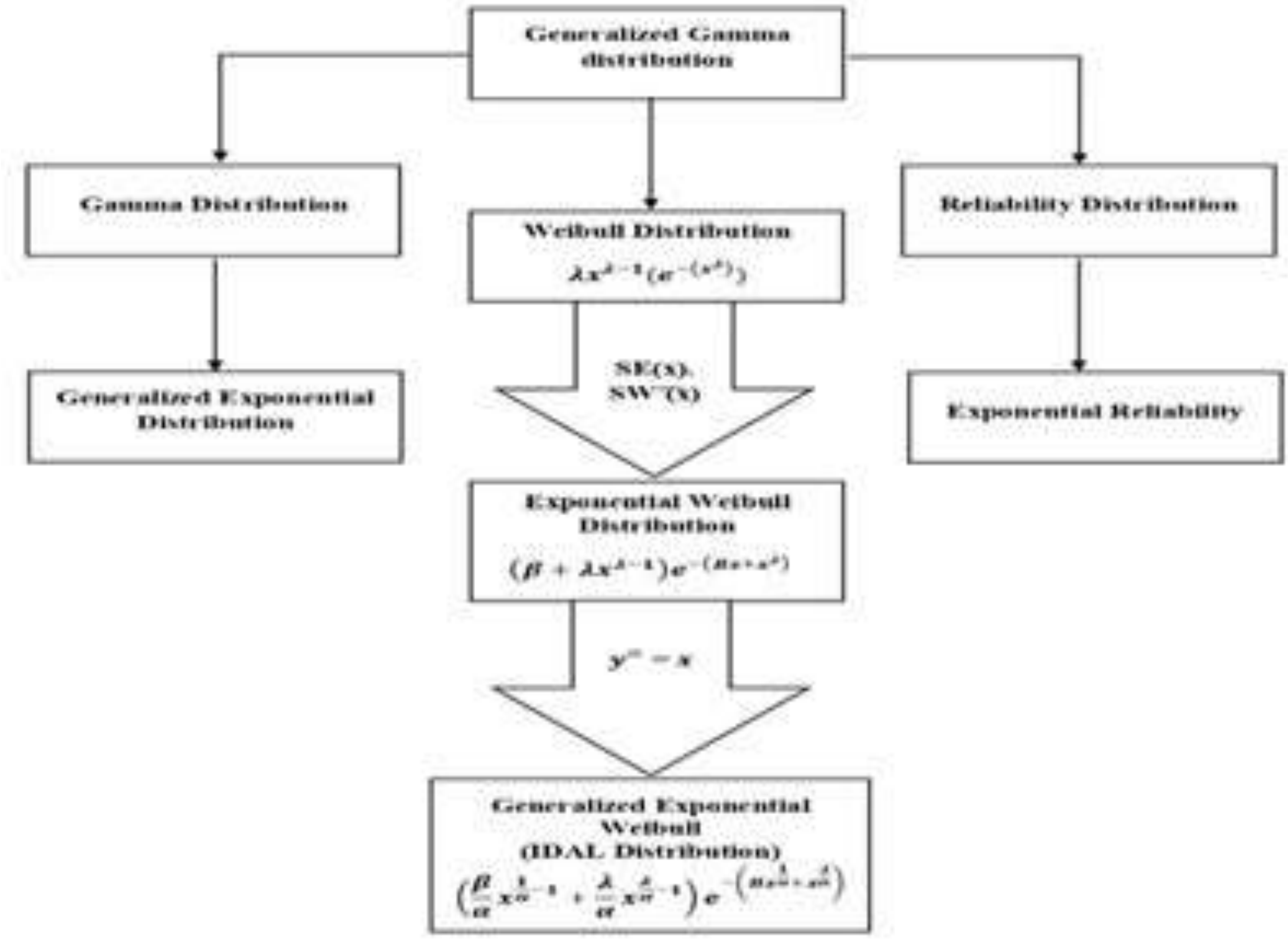

Figure 1. Relationship of IDAL Distribution with some distribution.

Hence, the probability density function (p.d.f.) of IDAL distribution is

$$f(x,\alpha,\beta,\lambda)=\left(\frac{\beta}{\alpha}x^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\right)e^{-\left(\beta x^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)} \quad x>0 \quad \dots(1)$$

$\Omega=\{(\alpha,\beta,\lambda):\alpha,\lambda,\beta>0$, where α and λ are the shape parameter and β is scale parameter.

The cumulative distribution function (C.D.F.) of the IDAL Distribution is:

$$F(x,\alpha,\lambda,\beta)=1-e^{-\left(\beta x^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)} \quad \dots(2)$$

The reliability function of the IDAL Distribution is:

$$R(x;\alpha,\lambda,\beta)=1-F(x;\alpha,\lambda\beta)=e^{-\left(Bx^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)} \quad x>0 \quad \dots(3)$$

The hazard rate function for IDAL distribution is:

$$h(x;\alpha,\lambda,\beta)=\frac{f(x;\alpha,\lambda,\beta)}{S(x;\alpha,\lambda,\beta)}$$

$$H(x,\alpha,B,\lambda)=\frac{\frac{B}{\alpha}x^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}e^{-\frac{\lambda}{\alpha}-1}}{1-e^{\left(Bx^{\frac{1}{\alpha}}+x^{\frac{\lambda}{\alpha}}\right)}} \quad \dots(4)$$

3. Application Some Estimation Methods for IDAL Distribution

This section will employ various estimating approaches for IDAL distribution, including the maximum likelihood method, the least squares method, and the proposed least squares method.

These strategies are utilized to forecast the optimal performance of the existing system as follows.

3.1. Maximum Likelihood Method (MLE)

The maximum likelihood method has a lot of important properties in comparison with other methods, so statisticians almost prefer it in many statistical applications [19]. Let $(x_1, x_2, \ldots, x_n)$; be random variable samples for sizes n. The probability density function for IDAL distribution will be

$$f(x;\lambda,\beta,\alpha)=\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}$$

$$Lf(x;\lambda,\beta,\alpha)=\prod_{I=1}^{n}\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} \quad \ldots(5)$$

Taking the natural loglikelihood function is:

$$lnLf(x;\lambda,\beta,\alpha)=\sum_{i=1}^{n} ln\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)-\sum_{i=1}^{n}\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right) \quad \ldots(6)$$

Taking the derivative for the obtained *Ln* function with respect to parameters (α), (β) and (λ) respectively and equating to zero, then

$$\frac{\partial \ln L}{\partial \alpha}=\sum_{i=1}^{n} -\frac{\frac{\beta}{\alpha^2}x_i^{\frac{1}{\alpha}-1}-\frac{\beta}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i-\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i-\frac{\lambda^2}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i}{\frac{1}{\alpha^2}\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{1}{\alpha}-1}\right)}+\sum_{i=1}^{n}\frac{1}{\alpha^2}\left(\beta x_i^{\frac{1}{\alpha}}+\lambda x_i^{\frac{1}{\alpha}}\right) \quad \ldots(7)$$

$$\frac{\partial \ln L}{\partial \beta}=\sum_{i=1}^{n}\frac{\frac{1}{\alpha}x^{\frac{1}{\alpha}-1}}{\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)}-\sum_{i=1}^{n}x_i^{\frac{1}{\alpha}} \quad \ldots(8)$$

$$=\sum_{i=1}^{n}\frac{x^{\frac{1}{2}-1}}{\left(Bx_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)}-\sum_{i=1}^{n}x_i^{\frac{1}{\alpha}} \quad \ldots(9)$$

$$\frac{\partial \ln L}{\partial \lambda}=\sum_{i=1}^{n}\frac{\frac{1}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i}{\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{1}{\alpha}-1}\right)}-\sum_{i=1}^{n}\frac{1}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\ln xi \quad \ldots(10)$$

$$\begin{bmatrix}\hat{\alpha}_{k+1}\\ \hat{B}_{k+1}\\ \hat{\lambda}_{k+1}\end{bmatrix}=\begin{bmatrix}\hat{\alpha}_{k}\\ \hat{B}_{k}\\ \lambda_{k}\end{bmatrix}-J^{-1}\begin{bmatrix}\frac{\partial \ln L}{\partial \alpha}\\ \frac{\partial \ln L}{\partial B}\\ \frac{\partial \ln L}{\partial \lambda}\end{bmatrix} \quad \text{Where } J \begin{bmatrix}\frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha\,\partial \beta} & \frac{\partial^2 \ln L}{\partial \alpha\,\partial \lambda}\\ \frac{\partial^2 \ln L}{\partial \beta\,\partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta^2} & \frac{\partial^2 \ln L}{\partial \beta\,\partial \lambda}\\ \frac{\partial^2 \ln L}{\partial \lambda\,\partial \alpha} & \frac{\partial^2 \ln L}{\partial \lambda\,\partial \beta} & \frac{\partial^2 \ln L}{\partial \lambda^2}\end{bmatrix}$$

$$\frac{\partial^2 \ln L}{\partial B^2}=\sum_{i=1}^{n}\frac{-\frac{1}{\alpha}x_i^{\frac{1}{\alpha}-1}\left(\frac{1}{\alpha}x_i^{\frac{1}{\alpha}-1}\right)}{\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)^2}=\sum_{i=1}^{n}\frac{-\left(x_i^{\frac{1}{\alpha}-1}\right)^2}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)^2} \quad \ldots(11)$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2}=\sum_{i=1}^{n}\frac{\frac{2\beta}{\alpha^3}x_i^{\frac{1+\lambda}{\alpha}-2}\ln x_i+\frac{\lambda\beta}{\alpha^4}x_i^{\frac{1+\lambda}{\alpha}-2}(\ln x_i)^2-\frac{1}{\alpha^2}x_i^{\frac{2\lambda}{\alpha}-2}}{\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)^2}-\sum_{i=1}^{n}\frac{1}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}(\ln x_i)^2 \quad \ldots(12)$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2}=\sum_{i=1}^{n}\Bigg[\frac{\beta}{\alpha^2}x_i^{\frac{1}{\alpha}-1}+\frac{\beta}{\alpha^3}x_i^{\frac{1}{\alpha}-1}\ln x_i+\frac{2\beta}{\alpha^3}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i+\frac{\beta}{\alpha^4}x_i^{\frac{1}{\alpha}-1}(\ln x_i)^2+\frac{\lambda}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}+\frac{2\lambda^2}{\alpha^3}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i+\frac{\lambda^3}{\alpha^4}x_i^{\frac{\lambda}{\alpha}-1}(\ln x_i)^2 + \left(\frac{\beta}{\alpha^2}x_i^{\frac{1}{\alpha}-1}\ln x_i\right)\left(-\frac{\beta}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}-\frac{\beta}{\alpha^2}x_i^{\frac{1}{\alpha}-1}\ln x_i-\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}-\frac{\lambda^2}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}\ln x_i\right)\Bigg]\Big/\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)^2 + \sum_{i=1}^{n}\frac{-2}{\alpha^3}\left(\beta x_i^{\frac{1}{\alpha}}\ln x_i+\lambda x_i^{\frac{\lambda}{\alpha}-1}\ln x_i\right)+\frac{1}{\alpha^2}\left(\frac{-\beta}{\alpha^2}x_i^{\frac{1}{\alpha}}(\ln x_i)^2-\frac{\lambda^2}{\alpha^2}x_i^{\frac{\lambda}{\alpha}-1}(\ln x_i)^2\right) \quad \ldots(13)$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \alpha}=\sum_{i=1}^{n}\frac{1}{\alpha^2}x_i^{\frac{1}{\alpha}}\,ln\,x_i=\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \quad \ldots(14)$$

$$\frac{\partial^2 \ln L}{\partial \lambda\, \partial \beta}=\sum_{i=1}^{n}\frac{-x_i^{\frac{1+\lambda}{\alpha}-2}\left(\frac{1}{\alpha^2}+\frac{1}{\alpha^3}\ln x_i\right)}{\left(\frac{\beta}{\alpha}x_i^{\frac{1}{\alpha}-1}+\frac{\lambda}{\alpha}x_i^{\frac{\lambda}{\alpha}-1}\right)^2}=\frac{\partial^2 \ln L}{\partial \beta\, \partial \lambda} \quad \ldots(14)$$

$$\frac{\partial^2 \ln L}{\partial \lambda\, \partial \alpha}=\frac{\sum_{i=1}^{n}\left[-\frac{2\lambda\beta}{\alpha^2}x_i^{\frac{1+\lambda}{\alpha}-2}\ln x_i-\frac{\lambda^2\beta}{\alpha^3}x_i^{\frac{1+\lambda}{\alpha}-2}(\ln x_i)^2-\frac{\lambda^2}{\alpha^2}x_i^{\frac{2\lambda}{\alpha}-2}\ln x_i+\frac{\beta}{\alpha^2}x_i^{\frac{1+\lambda}{\alpha}-2}\ln x_i+\frac{\beta\lambda}{\alpha^3}x_i^{\frac{1+\lambda}{\alpha}-2}(\ln x_i)^2\right]}{\left(\beta x_i^{\frac{1}{\alpha}-1}+\lambda x_i^{\frac{\lambda}{\alpha}-1}\right)^2}-\sum_{i=1}^{n}\frac{1}{\alpha^2}x_i^{\frac{\lambda}{\alpha}}\ln x_i\left[1+\frac{\lambda}{\alpha}\ln x_i\right]=\frac{\partial^2 \ln L}{\partial \alpha\, \partial \lambda} \quad \ldots(15)$$

3.1. Ordinary Least Square Error Method (O.L.S)

The least squares method is frequently employed in mathematical problems and parameter estimation [20]. The idea Let $x_1, x_2,\ldots, x_n$ represent a random sample of size n from a distribution function.

$$\sum_{i=1}^{n}\epsilon_i^2=\sum_{i=1}^{n}\left[1-e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}-\frac{\mathrm{i}-0.5}{\mathrm{n}}\right]^2 \quad \ldots(17)$$

$$\frac{\partial \sum \epsilon_i^2}{\partial \lambda}=x_i^{\frac{1}{\alpha}}\sum_{i=1}^{n}\frac{\mathrm{n}-\mathrm{i}+0.5}{\mathrm{n}}*e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}*x_i^{\frac{\lambda}{\alpha}}\ln x_i-\sum_{i=1}^{n}\frac{1}{\alpha}x_i^{\frac{\lambda}{\alpha}}\ln x_i\; e^{-2\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}$$

$$\frac{\partial \sum \epsilon_i^2}{\partial \alpha}=\frac{1}{\alpha^2}\sum_{i=1}^{n}\ln x_i\, e^{-2\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}*\left[\beta x_i^{\frac{1}{\alpha}}+\lambda x_i^{\frac{\lambda}{\alpha}}\right]+\frac{1}{\alpha^2}\sum_{i=1}^{n}\frac{\mathrm{n}-\mathrm{i}+0.5}{\mathrm{n}}\ln x_i\, e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}*\left[\beta x_i^{\frac{1}{\alpha}}+\lambda x_i^{\frac{\lambda}{\alpha}}\right] \quad \ldots(18)$$

$$\frac{\partial \sum \epsilon_i^2}{\partial \beta}=\sum_{i=1}^{n}\left[\frac{\mathrm{n}-\mathrm{i}+0.5}{\mathrm{n}}e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}x_i^{\frac{1}{\alpha}}-e^{-2\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}*x_i^{\frac{1}{\alpha}}\right]$$

$$\frac{\partial \sum \epsilon_i^2}{\partial \beta\, \partial \lambda}=\sum_{i=1}^{n}\left[\frac{\mathrm{n}-\mathrm{i}+0.5}{\mathrm{n}}e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}\left(\frac{-1}{\alpha}x_i^{\frac{1+\lambda}{\alpha}}\ln x_i\right)-e^{-2\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}*\left(-2\left(\frac{1}{\alpha}x_i^{\frac{1+\lambda}{\alpha}}\ln x_i\right)\right)\right]=\frac{\partial \sum \epsilon_i^2}{\partial \lambda\, \partial \beta} \quad \ldots(19)$$

$$\frac{\partial^2 \sum \epsilon_i^2}{\partial \alpha\, \partial \beta}=\sum_{i=1}^{n}e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}\frac{\ln x_i}{\alpha^2}x_i^{\frac{1}{\alpha}}*\left(-\beta x_i^{\frac{1}{\alpha}}+\lambda x_i^{\frac{\lambda}{\alpha}}-\beta x_i^{\frac{1}{\alpha}}+\lambda x_i^{\frac{\lambda}{\alpha}}\right)e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} + e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)}\left[\beta x_i^{\frac{1}{\alpha}}\left(\frac{\mathrm{n}-\mathrm{i}+0.5}{\mathrm{n}}\right)+\lambda x_i^{\frac{\lambda}{\alpha}}\left(\frac{\mathrm{n}-\mathrm{i}+0.5}{\mathrm{n}}\right)-\left(\frac{\mathrm{n}-\mathrm{i}+0.5}{\mathrm{n}}\right)\right]=\frac{\partial^2 \sum \epsilon_i^2}{\partial \beta\, \partial \alpha} \quad \ldots(20)$$

$$\frac{\partial^2 \sum \epsilon_i^2}{\partial\lambda\,\partial\alpha} = \sum_{i=1}^{n} \frac{1}{\alpha^2}(\ln x_i)\, x_i^{\frac{\lambda}{\alpha}} e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} \Big(- e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} \frac{\beta}{\alpha} x_i^{\frac{1}{\alpha}} \ln x_i - e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} \frac{\beta}{\alpha} x_i^{\frac{1}{\alpha}} \ln x_i - e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} * \frac{\lambda}{\alpha} x_i^{\frac{\lambda}{\alpha}} \ln x_i$$
$$+\ e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} +\ e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} \frac{\lambda}{\alpha} x_i^{\frac{\lambda}{\alpha}} \ln x_i + \frac{\beta}{\alpha} x_i^{\frac{1}{\alpha}} \ln x_i \left(\frac{\mathrm{n-i+0.5}}{\mathrm{n}}\right) - \frac{\mathrm{n-i+0.5}}{\mathrm{n}}$$
$$+ \frac{\lambda}{\alpha} x_i^{\frac{\lambda}{\alpha}} \ln x_i \left(\frac{\mathrm{n-i+0.5}}{\mathrm{n}}\right) - \frac{\lambda}{\alpha} \ln x_i \left(\frac{\mathrm{n-i+0.5}}{\mathrm{n}}\right) = \frac{\partial^2 \sum \epsilon_i^2}{\partial\alpha\,\partial\lambda} \qquad \ldots(21)$$

$$\frac{\partial \sum \epsilon_i^2}{\partial\beta^2} = \sum_{i=1}^{n} e^{-2\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} x_i^{\frac{2}{\alpha}} \left[\, 2e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} - \left(\frac{\mathrm{n-i+0.5}}{\mathrm{n}}\right)\right] \qquad \ldots(22)$$

$$\frac{\partial^2 \sum \epsilon_i^2}{\partial\lambda^2} = \sum_{i=1}^{n} e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} \frac{(\ln x_i)^2}{\alpha^2} x_i^{\frac{\lambda}{\alpha}} \left[-e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} + x_i^{\frac{\lambda}{\alpha}} \left(\frac{\mathrm{n-i+0.5}}{\mathrm{n}}\right) + \left(\frac{\mathrm{n-i+0.5}}{\mathrm{n}}\right)\right] \qquad \ldots(23)$$

$$\frac{\partial^2 \sum \epsilon_i^2}{\partial\alpha^2} = \sum_{i=1}^{n} \Bigg(e^{-2\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} \left(\frac{\beta}{\alpha^2} x_i^{\frac{1}{\alpha}} \ln x_i + \frac{\lambda}{\alpha^2} x_i^{\frac{\lambda}{\alpha}} \ln x_i\right)^2 + \left(\frac{\beta}{\alpha^2} x_i^{\frac{1}{\alpha}} \ln x_i + \frac{\lambda}{\alpha^2} x_i^{\frac{\lambda}{\alpha}} \ln x_i\right)^2 *$$
$$e^{-2\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} + e^{-2\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} \left(\frac{-2\beta}{\alpha^3} x_i^{\frac{1}{\alpha}} \ln x_i - \frac{\beta}{\alpha^4} x_i^{\frac{\lambda}{\alpha}} (\ln x_i)^2 - \frac{2\lambda}{\alpha^3} x_i^{\frac{\lambda}{\alpha}} \ln x_i \frac{-\lambda^2}{\alpha^4} x_i^{\frac{\lambda}{\alpha}} (\ln x_i)^2\right) + e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} \Bigg(\frac{\beta}{\alpha^2} x_i^{\frac{1}{\alpha}} \ln x_i +$$
$$\frac{\lambda}{\alpha^2} x_i^{\frac{\lambda}{\alpha}} \ln x_i\Bigg)^2 \left(\frac{\mathrm{n-i+0.5}}{\mathrm{n}}\right) - e^{-\left(\beta x_i^{\frac{1}{\alpha}}+x_i^{\frac{\lambda}{\alpha}}\right)} \left(\frac{-2\beta}{\alpha^3} x_i^{\frac{1}{\alpha}} \ln x_i - \frac{\beta}{\alpha^4} x_i^{\frac{\lambda}{\alpha}} (\ln x_i)^2 - \frac{2\lambda}{\alpha^3} x_i^{\frac{\lambda}{\alpha}} \ln x_i \frac{-\lambda^2}{\alpha^4} x_i^{\frac{\lambda}{\alpha}} (\ln x_i)^2\right) \left(\frac{\mathrm{n-i+0.5}}{\mathrm{n}}\right) \Bigg) \qquad \ldots(24)$$

3.1 The suggested Ordinary Least Square Error Method (S.O.L.S)

In section 2.2, we used the classical Least Square Error Method to estimate the parameters. In this section, we propose the least square error method via the reliability function instead of the cumulative distribution and empirical (c.d.f.) functions, as follows: $\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left[\hat{R}(t_i) - R(t_i)\right]^2$

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left[e^{-\left(\hat{\beta} t_{\mathrm{i}}^{\frac{1}{\hat{\alpha}}}+t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} \hat{} - e^{-\left(B t_{\mathrm{i}}^{\frac{1}{\alpha_1}}+t_i^{\frac{\lambda}{\alpha}}\right)} \right]^2$$

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left[-e^{-\left(\hat{\beta} t_{\mathrm{i}}^{\hat{\frac{1}{\hat{\alpha}}}}+t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} - \mathrm{C} \right]^2$$

$$\text{Where } C = e^{-\left(B t_{\mathrm{i}}^{\frac{1}{\alpha_1}}+t_i^{\frac{\lambda}{\alpha}}\right)}$$

$$\frac{\partial \sum_{i=1}^{n} \epsilon_i^2}{\partial\hat{B}} = c \sum_{i=1}^{n} t_i^{\frac{1}{\alpha}} e^{-\left(\hat{\beta} t_{\mathrm{i}}^{\frac{1}{\hat{\alpha}}}+t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} - \sum_{i=1}^{n} t_i^{\frac{1}{\alpha}} e^{-2\left(\hat{\beta} t_{\mathrm{i}}^{\frac{1}{\hat{\alpha}}}+t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)}$$

$$\frac{\partial \sum \epsilon_i^2}{\partial\hat{\lambda}} = C \sum_{i=1}^{n} \frac{1}{\hat{\alpha}} \left(t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right) (\ln t_i) e^{-\left(\hat{\beta} t_{\mathrm{i}}^{\frac{1}{\hat{\alpha}}}+t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} - \sum_{i=1}^{n} \sum_{i=1}^{n} \frac{1}{\hat{\alpha}} \left(t_i^{\frac{\hat{\lambda}}{\alpha}}\right) (\ln t_i)\, e^{-2\left(\hat{\beta} t_{\mathrm{i}}^{\frac{1}{\hat{\alpha}}}+t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)}$$

$$\frac{\partial \sum \epsilon_i^2}{\partial \hat{\alpha}} = \sum_{i=1}^{n} \frac{1}{(\hat{\alpha})^2} \ln t_i \cdot e^{-\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} \left[e^{-\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} \left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + \hat{\lambda} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right) - C\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + \hat{\lambda} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)\right]$$

$$\frac{\partial^2 \Sigma \epsilon_i^2}{\partial (\hat{\alpha})^2} = \sum_{i=1}^{n} 2e^{-\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} \left(\frac{\hat{\beta}}{(\hat{\alpha})^2} t_i^{\frac{1}{\hat{\alpha}}} \ln t_i + \frac{\hat{\lambda}}{(\alpha)^2} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} \ln t_i\right)^2 - \frac{1}{(\hat{\alpha})^2} \ln t_i \,.\, e^{-2\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} \left(\frac{2\hat{\beta}}{\hat{\alpha}} t_i^{\frac{1}{\hat{\alpha}}} + \frac{\hat{\beta}}{(\hat{\alpha})^2} t_i^{\frac{1}{\hat{\alpha}}} (\ln t_i) + \frac{2\hat{\lambda}}{\hat{\alpha}} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} \ln t_i + \right.$$

$$\left.\frac{(\hat{\lambda})^2}{(\hat{\alpha})^2} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} (\ln t_i) - e^{-\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} \frac{1}{(\hat{\alpha})^2} \ln t_i \left(\frac{\hat{\beta}}{(\hat{\alpha})^2} t_i^{\frac{1}{\hat{\alpha}}} + \frac{\lambda^2}{(\hat{\alpha})^2}\right)^2 . c\right) + e^{-\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} \frac{1}{(\hat{\alpha})^2} \ln t_i \left(\frac{-2\beta}{\hat{\alpha}} t_i^{\frac{1}{\hat{\alpha}}} + \frac{B}{(\hat{\alpha})^2} t_i^{\frac{1}{\hat{\alpha}}} (\ln t_i)^2 + \frac{2\hat{\lambda}}{\alpha^{\alpha}} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} - \right.$$

$$\left.\frac{\lambda^2}{(\hat{\alpha})^2} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} (\ln t_i)\right) . \mathrm{c}$$

$$\frac{\partial^2 \sum \epsilon_i^2}{\partial (\hat{B})^2} = C \sum_{i=1}^{n} -\left(t_i^{\frac{1}{\hat{\alpha}}}\right)^2 e^{-\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} + \sum_{i=1}^{n} 2\left(t_i^{\frac{1}{\hat{\alpha}}}\right)^2 e^{-2\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)}.$$

$$\frac{\partial^2 \sum \epsilon_i^2}{\partial \left(\hat{\lambda}\right)^2} = C \sum_{i=1}^{n} \frac{\ln t_i}{\hat{\alpha}} \left[\frac{1}{\hat{\alpha}} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} \ln t_i \, e^{-\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} + e^{-\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right] + \frac{\ln t_i}{\hat{\alpha}} e^{-2\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} \left[\frac{1}{\hat{\alpha}} \ln t_i - 2t_i^{\frac{1}{\hat{\alpha}}}\right]$$

$$\frac{\partial^2 \sum \epsilon_i^2}{\partial \hat{B} \hat{\lambda}} = C \sum_{i=1}^{n} -e^{-\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} t_i^{\frac{1}{\hat{\alpha}}} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} \ln t_i \frac{1}{\hat{\alpha}} + \sum_{i=1}^{n} t_i^{\frac{1}{\hat{\alpha}}} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} \ln t_i \frac{2}{\hat{\alpha}} e^{-2\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} = \frac{\partial^2 \sum \epsilon_i^2}{\partial \hat{\lambda} \hat{B}}$$

$$\frac{\partial^2 \sum \epsilon_i^2}{\partial \hat{B} \hat{\alpha}} = C \sum_{i=1}^{n} t_i^{\frac{1}{\hat{\alpha}}} \frac{\ln t_i}{(\hat{\alpha})^2} e^{-\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} \left(\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + \hat{\lambda} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right) - 1\right) + t_i^{\frac{1}{\hat{\alpha}}} \frac{\ln t_i}{(\hat{\alpha})^2} e^{-2\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} \left(2\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + \hat{\lambda} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right) + 1\right) = \frac{\partial^2 \sum \epsilon_i^2}{\partial \hat{\alpha} \hat{B}}$$

$$\frac{\partial^2 \sum \epsilon_i^2}{\partial \hat{\lambda} \hat{\alpha}} = C \sum_{i=1}^{n} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} \frac{\ln t_i}{(\hat{\alpha})^2} e^{-\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} \left(\left(\frac{\hat{\beta}}{\hat{\alpha}} t_i^{\frac{1}{\hat{\alpha}}} \ln t_i + \frac{\hat{\lambda}}{\hat{\alpha}} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} \ln t_i\right) - \left(1 + \frac{\hat{\lambda}}{\hat{\alpha}} \ln t_i\right)\right)$$

$$+ \sum_{i=1}^{n} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} \frac{\ln t_i}{(\hat{\alpha})^2} e^{-2\left(\hat{\beta} t_i^{\frac{1}{\hat{\alpha}}} + t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}}\right)} \left(2\left(\frac{\hat{\beta}}{\hat{\alpha}} t_i^{\frac{1}{\hat{\alpha}}} \ln t_i + \frac{\hat{\lambda}}{\hat{\alpha}} t_i^{\frac{\hat{\lambda}}{\hat{\alpha}}} \ln t_i\right) + \left(1 + \frac{\hat{\lambda}}{\hat{\alpha}} \ln t_i\right)\right) = \frac{\partial^2 \sum \epsilon_i^2}{\partial \hat{\alpha} \hat{\lambda}}$$

4. Simulation study

In this section, we will compare the performance of methods such as Maximum Likelihood, Least Square, and Suggested Least Square, using Monte-Carlo simulation to estimate the three parameters for the IDAL function distribution, based on mean squared errors $MSE(\varphi) = \sum_{i=1}^{n} (\varphi_i - \varphi)^2 / n$. Using unlike sample sizes n= (10, 20, 30, 50, 100) to represent small, moderate, and large sample sizes for IDAL function distribution and biltong 1000 replication [20]. We obtain the result shown in the following tables and figures.

Table (1) Shows the estimation for (α) parameter and MSE for all estimation methods

α	β	λ	n	*MLE*		*OLS*		SOLS		*best*
				$\hat{\alpha}_1$	mse_2	$\hat{\alpha}_2$	mse_2	$\hat{\alpha}_3$	mse_3	$best_{mse}$
0.25	1	0.1	10	0.247298	1.77E-03	0.266939	2.42E-03	0.270287	7.94E-04	3
0.25	1	0.1	20	0.25077	1.41E-05	0.251523	1.72E-05	0.24891	1.29E-05	3
0.25	1	0.1	30	0.249979	2.80E-07	0.25018	7.10E-08	0.249982	7.13E-08	2
0.25	1	0.1	50	0.250013	6.15E-10	0.249973	1.84E-09	0.249991	1.19E-09	1
0.25	1	0.1	100	0.250001	9.89E-12	0.250001	1.76E-11	0.249999	9.09E-12	3
0.25	1	0.3	10	0.273607	1.15E-03	0.258653	1.52E-03	0.232548	2.16E-03	1
0.25	1	0.3	20	0.252379	1.92E-05	0.252244	1.40E-05	0.249556	2.71E-05	2
0.25	1	0.3	30	0.250041	9.86E-08	0.249975	2.23E-08	0.249939	2.69E-07	2
0.25	1	0.3	50	0.249989	1.72E-09	0.249999	6.95E-10	0.249983	1.25E-09	2
0.25	1	0.3	100	0.249999	1.47E-11	0.250001	2.22E-11	0.250003	1.77E-11	1
0.25	2	0.1	10	0.258689	1.06E-03	0.258361	1.35E-03	0.260534	1.09E-03	1
0.25	2	0.1	20	0.250452	1.44E-05	0.250497	2.65E-05	0.246766	2.09E-05	1
0.25	2	0.1	30	0.249903	1.18E-07	0.250063	1.47E-07	0.24983	1.80E-07	1
0.25	2	0.1	50	0.250007	1.15E-09	0.249995	2.71E-09	0.250004	1.03E-09	3
0.25	2	0.1	100	0.249999	1.25E-11	0.250001	1.55E-11	0.250001	1.76E-11	1
0.25	2	0.3	10	0.259664	1.95E-03	0.268234	1.21E-03	0.229208	1.40E-03	2
0.25	2	0.3	20	0.248677	1.69E-05	0.250955	8.81E-06	0.249149	1.51E-05	2
0.25	2	0.3	30	0.250131	2.28E-07	0.25003	1.03E-07	0.249759	2.11E-07	2
0.25	2	0.3	50	0.249979	1.46E-09	0.24999	5.20E-10	0.249994	1.27E-09	2
0.25	2	0.3	100	0.250003	2.05E-11	0.250001	1.21E-11	0.249998	9.09E-12	3
0.75	1	0.1	10	0.736398	2.24E-03	0.758388	2.04E-03	0.769386	1.30E-03	3
0.75	1	0.1	20	0.74962	1.28E-05	0.747782	1.27E-05	0.749639	1.54E-05	2
0.75	1	0.1	30	0.749999	6.25E-08	0.749836	6.18E-08	0.749891	9.31E-08	2
0.75	1	0.1	50	0.750013	1.56E-09	0.750015	1.28E-09	0.750003	1.09E-09	3
0.75	1	0.1	100	0.750001	9.65E-12	0.750001	2.00E-11	0.749998	2.48E-11	1
0.75	1	0.3	10	0.742707	1.81E-03	0.762451	2.14E-03	0.745203	2.30E-03	1
0.75	1	0.3	20	0.751458	2.71E-05	0.750322	7.60E-06	0.749044	1.86E-05	2
0.75	1	0.3	30	0.750026	1.59E-07	0.750096	9.63E-08	0.74976	2.64E-07	2
0.75	1	0.3	50	0.750012	2.03E-09	0.750002	7.48E-10	0.750032	1.64E-09	2
0.75	1	0.3	100	0.750001	1.56E-11	0.749999	9.34E-12	0.750001	1.89E-11	2
0.75	2	0.1	10	0.746582	1.50E-03	0.752699	1.99E-03	0.744872	2.03E-03	1
0.75	2	0.1	20	0.751638	2.04E-05	0.749918	1.24E-05	0.749188	2.20E-05	2
0.75	2	0.1	30	0.750022	2.49E-07	0.749858	1.09E-07	0.749973	2.64E-07	2
0.75	2	0.1	50	0.749987	2.28E-09	0.749982	1.62E-09	0.749995	2.46E-09	2
0.75	2	0.1	100	0.750001	1.15E-11	0.750001	2.02E-11	0.749999	2.02E-11	1
0.75	2	0.3	10	0.743877	5.98E-04	0.732446	2.56E-03	0.750468	1.84E-03	1
0.75	2	0.3	20	0.748828	1.75E-05	0.749401	2.25E-05	0.750005	1.60E-05	3
0.75	2	0.3	30	0.749873	1.30E-07	0.749795	1.93E-07	0.750077	1.97E-07	1
0.75	2	0.3	50	0.749999	2.33E-09	0.750013	1.25E-09	0.749996	2.46E-09	2
0.75	2	0.3	100	0.750001	1.62E-11	0.750001	1.67E-11	0.750001	4.64E-12	3

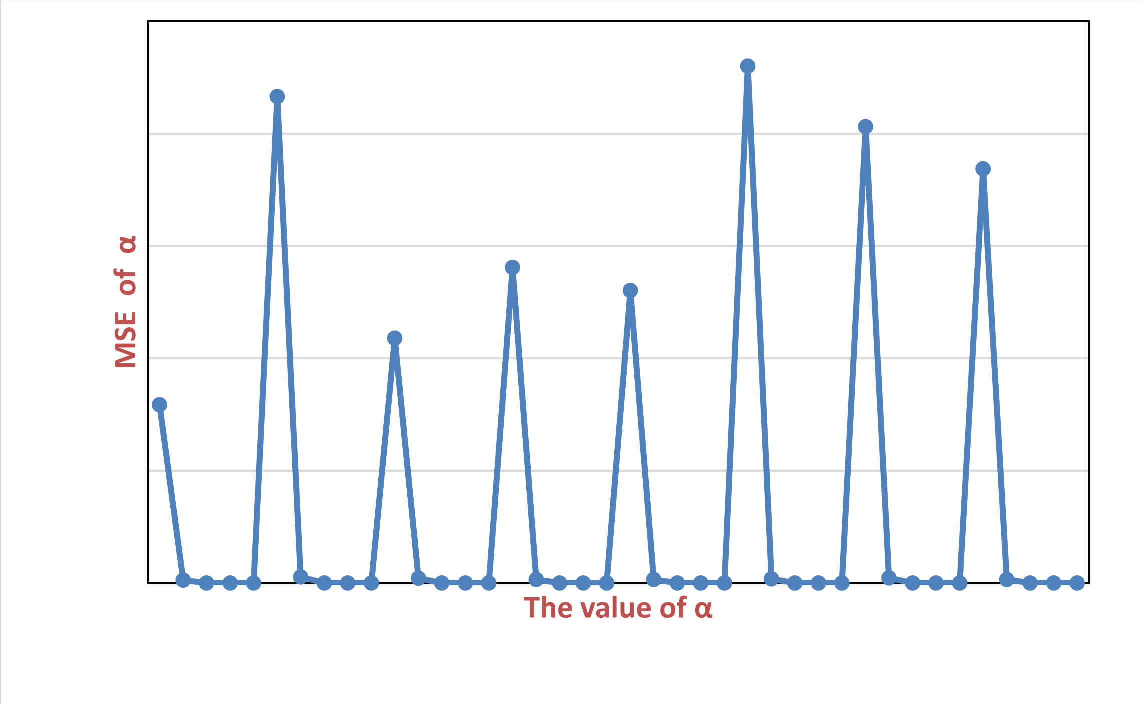

Figure 2: The minimum mean square error for (α)

Table (2) Shows the estimation for (β) parameter and MSE for all estimation methods

α	β	λ	n	MLE		*OLS*		*SOLS*		*best*
				$\hat{\beta}_1$	mse_2	$\hat{\beta}_2$	mse_2	$\hat{\beta}_3$	mse_3	$best_{mse}$
0.25	1	0.1	10	1.007481	1.90E-04	1.033028	1.73E-03	1.011581	1.78E-03	1
0.25	1	0.1	20	0.999759	2.11E-05	1.000234	1.12E-05	0.998912	1.95E-05	2
0.25	1	0.1	30	1.000018	7.93E-08	1.000134	2.38E-07	1.000169	1.82E-07	1
0.25	1	0.1	50	1.000001	1.21E-09	0.999989	1.31E-09	1.000003	7.74E-10	3
0.25	1	0.1	100	1.000001	1.04E-11	1.000003	3.51E-11	1.000001	2.24E-11	1
0.25	1	0.3	10	0.995401	5.13E-04	1.00588	1.53E-03	0.980183	2.04E-03	1
0.25	1	0.3	20	1.001783	1.52E-05	0.99916	1.73E-05	0.99932	1.81E-05	1
0.25	1	0.3	30	1.000103	7.64E-08	1.000011	1.39E-07	0.999772	1.89E-07	1
0.25	1	0.3	50	0.999996	1.89E-09	0.999988	2.41E-09	1.000001	2.49E-09	1
0.25	1	0.3	100	1.000001	1.80E-11	1.000002	1.85E-11	0.999999	1.64E-11	3
0.25	2	0.1	10	2.015462	6.92E-04	2.018167	1.63E-03	1.983024	1.36E-03	1
0.25	2	0.1	20	1.997942	1.78E-05	1.999483	1.83E-05	2.000792	2.71E-05	1
0.25	2	0.1	30	1.99992	1.70E-07	1.99982	1.95E-07	1.999945	1.04E-07	3
0.25	2	0.1	50	1.999982	1.36E-09	1.999986	2.71E-09	1.999994	7.28E-10	3
0.25	2	0.1	100	2.000001	1.60E-11	1.999999	1.38E-11	2.000001	2.35E-11	2
0.25	2	0.3	10	2.011011	1.66E-03	2.00258	1.42E-03	2.00622	5.06E-04	3
0.25	2	0.3	20	1.999699	1.32E-05	2.002227	2.77E-05	2.000017	2.17E-05	1
0.25	2	0.3	30	2.000038	6.92E-08	1.999934	6.67E-08	2.000084	1.63E-07	2
0.25	2	0.3	50	1.999997	1.30E-09	2.000013	1.95E-09	2.000012	2.15E-09	1
0.25	2	0.3	100	1.999999	1.00E-11	2.000001	1.53E-11	1.999999	2.22E-11	1
0.75	1	0.1	10	1.016564	8.11E-04	0.999505	2.85E-03	0.980665	1.72E-03	1
0.75	1	0.1	20	0.999625	2.19E-05	1.002536	2.78E-05	0.999693	1.01E-05	3
0.75	1	0.1	30	0.999787	1.97E-07	1.000085	9.85E-08	0.999972	7.86E-08	3
0.75	1	0.1	50	0.999995	1.20E-09	1.000001	1.85E-09	1.000004	8.55E-10	3
0.75	1	0.1	100	0.999998	1.70E-11	1.000001	7.38E-12	1.000001	1.73E-11	2
0.75	1	0.3	10	1.004485	1.64E-03	1.00975	1.68E-03	0.987514	1.20E-03	3
0.75	1	0.3	20	0.998957	1.62E-05	0.999333	1.54E-05	1.000929	1.34E-05	3
0.75	1	0.3	30	0.99999	2.19E-07	1.00008	1.15E-07	1.000041	7.50E-08	3
0.75	1	0.3	50	0.999995	1.62E-09	0.999997	5.77E-10	0.999995	2.42E-09	2
0.75	1	0.3	100	1.000001	1.87E-11	1.000001	6.09E-12	0.999999	1.56E-11	2
0.75	2	0.1	10	1.983691	1.46E-03	2.008149	3.02E-03	1.983006	2.45E-03	1
0.75	2	0.1	20	2.000338	1.96E-05	1.999618	1.09E-05	1.999927	9.76E-06	3
0.75	2	0.1	30	1.999859	1.22E-07	1.999886	6.56E-08	1.999958	1.15E-07	2
0.75	2	0.1	50	1.999996	1.04E-09	2.000016	2.12E-09	1.999999	1.76E-09	1
0.75	2	0.1	100	2.000001	9.96E-12	1.999999	1.59E-11	2.000002	1.47E-11	1
0.75	2	0.3	10	2.002361	1.67E-03	1.96393	3.16E-03	1.989749	4.32E-04	3
0.75	2	0.3	20	1.999778	7.89E-06	2.001089	1.97E-05	1.998315	1.23E-05	1
0.75	2	0.3	30	2.000051	9.98E-08	2.000022	2.48E-07	2.000124	1.51E-07	1
0.75	2	0.3	50	2.000017	2.05E-09	2.000002	2.10E-09	2.000011	1.36E-09	3
0.75	2	0.3	100	1.999999	8.67E-12	1.999998	2.31E-11	2.000001	1.53E-11	1

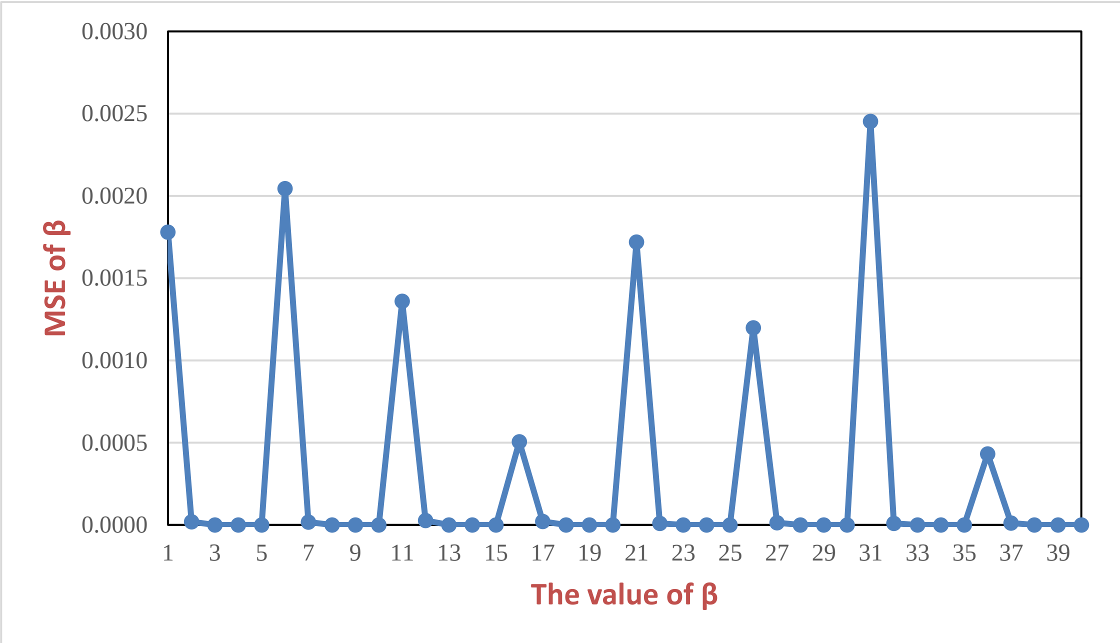

Figure (3) The minimum mean square error for (β)

Table (3) Shows the estimation for (λ) parameter and MSE for all estimation methods

α	β	λ	n	MLE		OLS		SOLS		best
				$\hat{\lambda}_1$	mse_2	$\hat{\lambda}_2$	mse_2	$\hat{\lambda}_3$	mse_3	$best_{mse}$
0.25	1	0.1	10	9.29E-02	1.80E-03	0.133375	1.76E-03	0.126672	2.42E-03	2
0.25	1	0.1	20	0.105148	4.58E-05	0.103766	3.01E-05	0.107752	7.96E-05	2
0.25	1	0.1	30	0.104534	3.10E-05	0.10425	2.69E-05	0.105824	4.24E-05	2
0.25	1	0.1	50	0.104586	2.76E-05	0.104339	2.71E-05	0.105758	4.17E-05	2
0.25	1	0.1	100	0.103436	1.60E-05	0.105264	3.49E-05	0.104655	2.68E-05	1
0.25	1	0.3	10	0.295757	1.52E-03	0.325449	1.77E-03	0.306053	1.88E-03	1
0.25	1	0.3	20	0.304852	7.32E-05	0.301513	3.23E-05	0.303518	2.92E-05	3
0.25	1	0.3	30	0.306404	4.84E-05	0.304003	2.19E-05	0.306571	4.99E-05	2
0.25	1	0.3	50	0.303178	1.48E-05	0.303351	2.04E-05	0.305671	4.32E-05	1
0.25	1	0.3	100	0.304966	3.69E-05	0.304417	3.17E-05	0.304748	3.42E-05	2
0.25	2	0.1	10	0.115314	2.86E-03	9.93E-02	1.32E-03	0.11057	1.10E-03	3
0.25	2	0.1	20	0.104987	3.27E-05	0.1061	7.73E-05	0.106313	7.22E-05	1
0.25	2	0.1	30	0.104809	3.17E-05	0.105959	4.34E-05	0.104692	3.30E-05	1
0.25	2	0.1	50	0.105692	4.22E-05	0.105831	4.37E-05	0.104884	3.23E-05	3
0.25	2	0.1	100	0.104269	2.44E-05	0.10646	4.99E-05	0.10522	3.91E-05	1
0.25	2	0.3	10	0.325714	1.12E-03	0.308487	1.16E-03	0.325792	1.68E-03	1
0.25	2	0.3	20	0.306653	6.30E-05	0.302453	1.89E-05	0.300685	1.66E-05	3
0.25	2	0.3	30	0.305049	3.44E-05	0.304908	3.51E-05	0.304536	3.12E-05	3
0.25	2	0.3	50	0.30449	2.90E-05	0.304159	2.82E-05	0.304865	3.10E-05	2
0.25	2	0.3	100	0.3032	1.92E-05	0.305813	4.05E-05	0.304247	2.60E-05	1
0.75	1	0.1	10	0.118926	1.80E-03	9.24E-02	1.30E-03	0.101671	1.29E-03	3
0.75	1	0.1	20	0.100527	1.61E-05	0.104412	3.39E-05	0.105486	4.48E-05	1
0.75	1	0.1	30	0.106275	4.80E-05	0.104337	3.29E-05	0.105624	4.51E-05	2
0.75	1	0.1	50	0.103564	2.09E-05	0.104596	2.69E-05	0.105014	2.93E-05	1
0.75	1	0.1	100	0.104901	2.97E-05	0.104989	2.96E-05	0.104977	3.30E-05	2
0.75	1	0.3	10	0.302755	1.83E-03	0.298556	2.15E-03	0.303671	2.40E-03	1
0.75	1	0.3	20	0.304031	4.23E-05	0.304693	3.59E-05	0.302964	1.98E-05	3
0.75	1	0.3	30	0.305799	3.92E-05	0.303944	2.34E-05	0.305112	3.35E-05	2
0.75	1	0.3	50	0.304845	3.43E-05	0.303933	2.52E-05	0.30577	4.28E-05	2
0.75	1	0.3	100	0.305384	3.43E-05	0.304976	3.66E-05	0.304306	2.34E-05	3
0.75	2	0.1	10	0.103747	5.23E-04	7.61E-02	1.53E-03	9.04E-02	3.35E-03	1
0.75	2	0.1	20	0.105322	5.80E-05	0.105979	5.43E-05	0.106193	5.69E-05	2
0.75	2	0.1	30	0.105117	3.53E-05	0.104873	2.85E-05	0.105693	4.47E-05	2
0.75	2	0.1	50	0.105616	3.90E-05	0.105165	3.59E-05	0.105735	4.31E-05	2
0.75	2	0.1	100	0.103624	2.18E-05	0.103816	2.38E-05	0.106639	5.39E-05	1
0.75	2	0.3	10	0.308371	6.63E-04	0.293145	1.34E-03	0.307612	1.38E-03	1
0.75	2	0.3	20	0.306143	4.50E-05	0.30431	4.58E-05	0.302809	2.49E-05	3
0.75	2	0.3	30	0.304651	3.16E-05	0.304703	2.78E-05	0.305544	3.78E-05	2
0.75	2	0.3	50	0.305978	4.08E-05	0.30463	2.67E-05	0.305326	3.43E-05	2
0.75	2	0.3	100	0.302923	1.27E-05	0.306412	4.93E-05	0.305313	3.90E-05	1

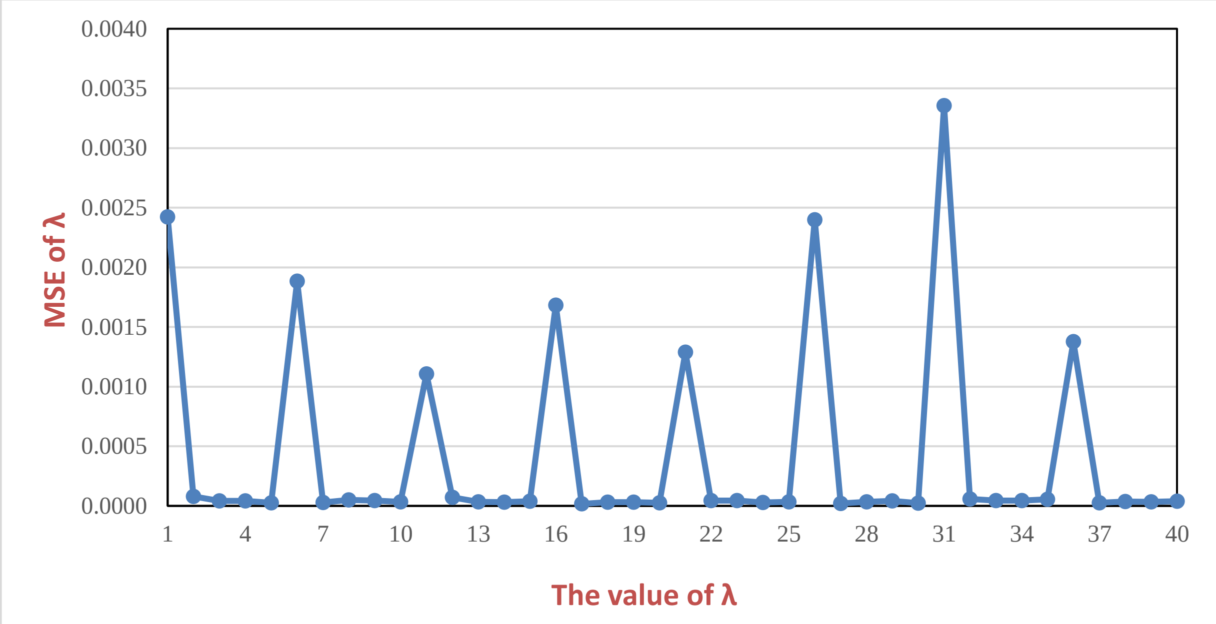

Figure 4: The minimum mean square error for (λ)

5. Numerical results

The simulations for the estimators of the parameters model of the IDAL distribution gave a significant numerical result as it was tested with sample sizes of 10, 20, 30, 50, and 100. It can be drawn from the obtained results in Tables 1, 2, and 3 that the least squares method estimates provide the best performance in all estimations, with mention that the MLE in β is considered the best method according to the comparison indicators. It can be noted from Figures 2, 3, and 4 that the best estimation result was (1.66E-05) for the simulation case ($\alpha = 0.25, \beta = 2, \lambda = 0.3, n = 20$) using the estimation method (OLS), and the best estimation result was (1.47E-11) for the simulation case ($\alpha = 0.75, \beta = 2, \lambda = 0.1, n = 100$) using the (MLE) estimation method.

6. Conclusion

We have suggested a novel distribution and investigated its mathematical properties. The classical estimating methods have been addressed in this work to estimate the three unknown parameters of the proposed distribution. In comparison between the employed methods, it has been mentioned that the least squares method is considered the most effective method for estimating the parameters in all implemented cases, as it has the lowest mean square error (MSE).

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