Dependent Elements of Biadditive Mappings on Semiprime Rings

Abdul-Rahman H. Majeed* and Auday Hekmat Mahmood**
*Department of Mathematic, College of Science, University of Baghdad.
**Department of Mathematic, College of Education, University of Al-Mustansiriyah.

Abstract

In this paper; we introduce the concepts of dependent element and free action associated to a biadditive mapping. We shall investigate some properties of dependent element of various mapping related to Bicentralizer on a semiprimerings. Also we identify various situations where these maps are free action.

Keywords: Dependent elements, free action mappings, Semiprime rings, left (right) Bicentralizer.

1- Introduction

The concepts of dependent element and free action are intimately connected. The notion of dependent element originated from generalized the results concerning free action mapping (defined on abelian von Neumann), that introduced by Murray and von Neumann [9] and von Neumann [11].

The first using of dependent element was implicitly by Kallman [5], when he generalized the notion of free action of automorphisms of von Neumann algebra, not necessarily abelian. In order to generalized the concept freely acting automorphisms to C^*-algebras, Choda, Kashahara and Nakamoto in [3] introduced the notion of dependent element associated to Automorphisms.

The fact that C^*-algebras and von Neumann algebras are Semiprime rings, and that a von Neumann algebra is prime if and only if its center consists of scalar multiples of identity, was the main reason that motivated several authors to extend the notion of dependent element of mapping and free action on C^*-algebras and von Neumann algebras, and to study these notions in the context of semiprime and prime rings. The initial results on this subject obtained in 1998 by Laradji and Thaheem [7], a number of results which are state in [3] have been generalized to semiprime rings. In the recent years, an important studies have been made by Vukman and Kosi- UIbI [12], Vokman [14], and A. Muhammad and S. Muhammad[10], which are concerning the dependent elements of various mapping related to Automorphisms, derivations, (α,β)-derivations, and generalized derivations of semiprime rings.

Throughout, unless otherwise mentioned, R represents an associative ring with center $Z(R)$and extended centroid C.A ring R is n-torsion free, where n is an integer, in the case $nx=0$, $x \in R$ implies $x=0$. In the sequel we shall usually write $[x, y]$ for the commutator $xy-yx$. We shall make extensive use of the commutator identities $[xz, y] = x[z, y] + [x, y]z$,and$[x, yz] = y[x, z] + [x, y]z$. Recall that a ring R is prime in case $aRb = (0)$ implies that either $a=0$ or $b=0$,equivalently, if the product of any two nonzero ideals of R is nonzero. A ring $R$is said to besemiprime if $a Ra=(0)$ implies $a=0$, Equivalently, if R has no nonzero nilpotent ideals. So it is easy to see that in semiprime rings, there are no nonzero nilpotent Dependent elements.

An additive mapping d: $R \longrightarrow R$ is called a derivation if $d(xy)= d(x) y+x d(y)$ holds for all $x,y \in R$. A biadditive mapping $D:R\times R\longrightarrow R$ is called a Biderivation if it is a derivation in each argument: that is for every $x \in R$ the maps $y \longrightarrow G(x, y)$ and $y \longrightarrow G(y, x)$ are derivations on R.

A mapping $B: R\times R\longrightarrow R$ is called Symmetric if $B(x, y) = B(y, x)$, for all pairs $x,y \in R$. A biadditive mapping $T: R \longrightarrow R$ is called a left (right)Bicentralizer if $T(xz, y)= T(x, y)z$ and $T(x, yz)=T(x, y)z$ ($T(xz, y)= xT(z, y)$ and $T(x, yz)=yT(x, z)$),holds for all $x,y,z \in R$. On the other hand, T is called Bicentralizer in case T is both left and right Bicentralizer. An ideal I of a ring R is said to be essential if it has nonzero intersection with any nonzero ideal of R. It should be recalled that the right annihilator $r(I)$ of I in R is a totality of all $x \in R$ such that $Ix=0$. Accordingly, the left annihilator $\ell(I)$ is a set of all $x \in R$ such that

$xI=0$. The intersection $ann(I) = r(I)\cap \ell(I)$ is called an annihilator of I in R(see[6]).

In this paper

We introduce the concept of the dependent elements and free action associated to biadditive mapping defined on R.

Definition (1.1):

Let R be a ring. An element $a\in R$is called dependent element of a biadditivemapping $F: R\times R\longrightarrow R$ if $F(x, y)a = ayx$ holds for all $x,y \in R$. The collection of all dependent elements of F denotes by $\mathcal{D}(F)$.

Definition (1.2):

Let R be a ring. A mapping F: $R\times R\longrightarrow R$ is called a free action in case zero is the only dependent element.

***2.* Preliminary results:**

We begin with the following known results which will be used extensively to prove our main results.

Lemma (2.1):[2]

Let R be a 2-torsion free semiprime ring and let $a,b \in R$. If for all $x\in R$, the relation $axb+bxa=0$ hold, then $axb=bxa=0$isfulfilled for all $x \in R$.

Lemma (2.2):*[13]*

Let R be a prime ring with extended centroid C, and let $a,b \in R$ besuch that $axb=bxa$ holds for all $x \in R$. If $a\neq 0$, then there exist $\lambda \in C$ such that $b= \lambda a$.

Lemma (2.3):*[6]*

Let I be an ideal of a semiprime ring R. Then $ann(I) = r(I)=\ell(I)$.

Lemma (2.4):*[6]*

Let I be an ideal of asemiprime ring R. then $I \oplus ann(I)$ is an essential ideal of R.

Lemma (2.5): *[8]*

Let R be a simeprime ring, I an ideal of R, then I is a simeprime subring of R and $Z(I)\subseteq Z(R)$.

Lemma (2.6):*[1]*

If R is a semiprime ring and I is an ideal of R, then $I \cap r(I)=0$.

Lemma 2.7:*[4]*

Let R be a simeprime ring, and $a \in R$ satisfies $a[a, x] =0$, for all $x \in R$, then $a \in Z(R)$.

We need to introduce the following Lemma.

Lemma (2.8):

Let R be a ring with identity. Then a symmetric biadditivemapping $F:R\times R\longrightarrow R$ is a symmetric left (right) Bicentralizer if and only if F is of the form $F(x, y) = ayx$**(**$F(x, y)= xya$**)** for some fixed element$a \in R$.

Proof:

Suppose $F$is a Symmetric left Bicentralizer: $F(xz, y)= F(x, y) z= F(1.x, y)= F(1, y) xz$.
$=F(1.y, 1) xz = ayxz$, where a stands for $F(1, 1)$

Hence $F(xz, y) = ayxz$ for all $x,y,z \in R$.
Taking $z =1$ leads to:

$$F(x, y) =ayx \text{ , for all } x,y \in R.$$

Conversely, suppose $F(x, y) = ayx$ for all $x,y \in R$ then

$$F(xz, y)= ayxz= (ayx)z = F(x, y) z$$

Hence F is a Symmetric left Bicentralizer.
In similar arguments, we can prove that$F$ is a Symmetric right Bicentralizer if and only if$F(x, y)=xya$. ■

Now, we introduce the following definition which will be use in some main results.

Definition (2.9):

Let Ube a subring of R, and let S,T: $U\times U\longrightarrow R$, be a symmetric biadditive mappings, then the pair (T, S) is called double Bicentralizer of U if $T$is a left Bicetralizer and S is a right Bicentralizer, as well as they satisfy a balanced condition $zT(x, y) = S(y, z)x$, for all $x,y,z \in U$.

Example (2.10):

Let Z be the ring of integers, and $R=M_2(Z)$ be the set of all matrices of order 2, then R is a commutative ring with respect to the usual operation of addition and matrix multiplication, also let:

$$U=\left\{\begin{pmatrix} a & a \\ a & a \end{pmatrix}, a \in Z\right\}.$$

Let S, T: $U\times U\longrightarrow R$ be two bidditive mappings defined by:

$$T(\begin{pmatrix} a & a \\ a & a \end{pmatrix}, \begin{pmatrix} b & b \\ b & b \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ ab & ab \end{pmatrix}$$
$$S(\begin{pmatrix} a & a \\ a & a \end{pmatrix}, \begin{pmatrix} b & b \\ b & b \end{pmatrix}) = \begin{pmatrix} 0 & ab \\ 0 & ab \end{pmatrix}$$

Then (T, S) is double Bicentralizer of U. ■

Finally, we see it's suitable to introduce the following Lemma.

Lemma (2.11):
Let R be a semiprime ring. Then every double Bicentralizer(T, S) induced a Symmetric Biderivation D defined by $D(x, y)= T(x, y)- S(x, y)$.
Proof: We have

$$D(x, y)= T(x, y)- S(x, y), \text{ for all } x,y\in R.$$

Putting xz instead of x in the above relation, we get:

$$\begin{aligned} D(xz, y) &= T(x, y)z-xS(z, y) \\ &= (T(x, y)- S(x, y))z+ x(T(y, z)- S(z, y)) \\ &= D(x, y)z +xD(z, y), \text{ for all } x,y,z\in R. \end{aligned}$$

3. **Main Results**:
Theorem (3.1):
Let R be a simeprime ring and $T: R\times R\longrightarrow R$ be a left Bicentralizer, $a \in R$. Then $a \in \mathcal{D}(T)$ if and only if $a \in Z(R)$ and $T(a, y)=ay$ holds for all $y \in R$.

Proof:
Let $a \in \mathcal{D}(T)$, then
$T(x, y)a = ayx$, for all $x, y \in R$. .... (1)
We consider
$$T(xa^2\omega, y)= T(x, y)\, a^2\omega= ayxa\omega = T(xa\omega, y)a = T(x, y)a\omega a$$

Therefore
$T(x, y)a\, [a, \omega]=0$, for all $x, y,\omega \in R$. .... (2)

According to (1), the above relation becomes:
$ayx\, [a, \omega]=0$, for all $x, y, \omega \in R$. .... (3)

Replacing x by a and y by $[a, \omega]\, y$, we get:

$$a[a, \omega]\, y\, a[a, \omega]=0, \text{ for all } y, \omega \in R.$$

Using the semiprimeness of R leads to:

$$a[a, \omega]=0, \text{for all } \omega \in R.$$

Consequently by Lemma (2.7), we conclude that $a \in Z(R)$. So $T(a\omega, y)=T(\omega a, y)$, for all $y, \omega \in R$. That is

$$T(a, y)\, \omega = T(\omega, y)\, a=ay\omega.$$
Whence $(T(a, y) - ay)\omega =0$, for all $y, \omega \in R$, then using semiprimeness of R, we get:

$$T(a, y) = ay, \text{ for all } y \in R.$$

Conversely, suppose that$a \in Z(R)$ and $T(a, y)=ay$ holds for all $y \in R$, then:

$$T(xa, y)= T(ax, y) \text{ , for all } x, y \in R.$$
$$T(x, y)\, a= T(a, y)\, x = ayx, \text{ for all } x, y\in R.$$

Hence $a \in \mathcal{D}(T)$.■

Corollary (3.2):
If R is a simeprime ring with $Z(R)= \{0\}$, then the left Bicentralizer$T: R\times R\longrightarrow R$is free action.

Theorem (3.3):
Let R be a simeprime ring, and $S: R\times R\longrightarrow R$be a right Bicentralizer, $a \in \mathcal{D}(S)$, then $a \in Z(R)$.

Proof:
Since $a \in \mathcal{D}(S)$, then we have:
$S(x, y)a = ayx$, for all $x, y\in R$.(1)
Replacing x by xzin (1), we get:
$$xS(z, y)a = ayxz, \text{ for all } x, y\in R.$$
According to (1), the above relation reduces to:
$$xayz = ayxz, \text{ for all } x, y\in R.$$
That is
$$[x, ay]\, z=0, \text{ for all } x, y\in R.$$

Using the identity $[x, yz] = [x, y]z + y[x, z]$, we obtain:
$$[x, a]y + a[x, y]=0, \text{ for all } x, y\in R.$$

Setting $x=a$ in the above relation implies that:
$$a[a, y]=0, \text{ for all } y\in R.$$
Using Lemma (2.7), we get the assertion of the Theorem. ■

Theorem (3.4):
Let R be a prime ring and $T: R\times R\longrightarrow R$ be a Symmetric left Bicentralizersuch that $T(x, y)\neq yx$, then T is a free action.

Proof:
Let $a \in \mathcal{D}(T)$, then $a \in Z(R)$ by Theorem (3.1). So we have:

$$T(x, y)a = ayx = yxa \text{ , for all } x, y\in R.$$
That is
$$(T(x, y)-yx\,)a=0 \text{ , for all } x, y\in R.$$
Right multiplication of the above relation by ω gives:
$$(T(x, y)-yx)\omega\, a=0 \text{ , for all } x, y\in R.$$
Since $T(x, y)\neq yx$, and R is prime ring, we get $a=0$. So T is free action.■

Theorem (3.5):

Let R be a semiprime ring and $T: R\times R\longrightarrow R$ be a left Bicentralizer, then $\mathcal{D}(T)$ is a semiprime subring of $Z(R)$.

Proof: Let $a,b \in \mathcal{D}(T)$, then by Theorem (3.1) we have $a,b\in Z(R)$, $T(a, y) = ay$ and $T(b, y) = by$, for all $y \in R$.

Consequently, $a\text{-}b \in Z(R)$, $ab \in Z(R)$.

Also

$$T(a-b, y) = T(a, y) - T(b, y) = ay - by = (a-b)y$$
$$T(ab, y) = T(a, y)b = ayb = aby$$

Again, by Theorem (3.1) we have $a\text{-}b \in \mathcal{D}(T)$ and $ab \in \mathcal{D}(T)$.

Hence $\mathcal{D}(T)$ is a subring of $Z(R)$.■

Corollary (3.6):

Let R be a semiprime ring, and $T:R\times R\longrightarrow R$ be a Bicentralizer, then $\mathcal{D}(T)$ is an ideal of R.

Proof:

Since $\mathcal{D}(T)$ is a semiprime subring of $Z(R)$, we have only to show that $ar \in \mathcal{D}(T)$, for all $a \in \mathcal{D}(T)$ and $r \in R$.

$$T(x, y)ar = ayxr = yxar = yx\, T(a, r) = T(yxa, r) = T(yx, r)a = aryx$$

Whence $ar \in \mathcal{D}(T)$.■

Theorem (3.7):

Let R be a semiprime ring and $T:R\times R\longrightarrow R$ be a Bicentralizer. Then there exist an ideals U and V of R such that:

(1) $U\oplus V$ is an essential ideal of R with $U\cap V = \{0\}$.

(2) $T(V, V)\subset V$.

(3) T is free action on V.

Proof:

By Corollary (3.6), we have $\mathcal{D}(T)$ is an ideal of R.

Let $U= \mathcal{D}(T)$ and $V=ann(U)$, then V is an ideal of R and $U\cap V = \{0\}$ by Lemma (2.6), also we have $U\oplus V$ is an essential ideal of R by Lemma (2.4).

Now, let $x,y \in V = ann(U)$, then $xa = ax = 0$ and $ay = ya = 0$, for all $a \in U$, moreover $T(x, y)a = ayx = 0$, hence $T(V, V)\subset V$.

Finally, by Lemma (2.5) we have V is a semiprime subring of R, and $Z(V)\subseteq Z(R)$.

Now, let $c \in V$ be a dependent element of the restriction of T on V, then by Theorem (3.1) we have $c \in Z(V)\subseteq Z(R)$. Moreover

$$T(c, u) = cu, \text{ for all } u \in U.$$

Replacing u by ru in the above relation, we get:

$$(T(c, r)-cr)u=0, \text{ for all } r \in R \text{ and } u \in U.$$

The semiprimeness of U leads to $T(c, r) = cr$, for all $r \in R$, hence $c \in \mathcal{D}(T)=U$.

Since $U\cap V = \{0\}$, we have $c=0$. Hence T is free action.■

Theorem (3.8):

Let T be a right Bicentralizer of a semiprime ring R, then φ: $R\times R\longrightarrow R$ defined by $\varphi(x, y) = [T(x, y), x]$, for all $x,y \in R$ is free action.

Proof:

Let $a \in \mathcal{D}(\varphi)$, then $\varphi(x, y)a = ayx$, for all $x,y \in R$, that is:

$[T(x, y), x]a=ayx$, for all $x,y\in R$. .... (1)

The linearization of (1) with respect to x gives:

$[T(x, y), \omega]a + [T(\omega, y), x]a = 0$, for all $x,y,\omega \in R$. .... (2)

Putting $a\omega$ for ω in (2), we get:

$$a[T(x, y), \omega]a + [T(x, y),a]\omega a + a[T(\omega, y), x]a + [a ,x]T(\omega, y)a = 0, \text{ for all } x,y,\omega \in R.$$

In view of (2), the above relation reduces to:

$[T(x, y),a]\omega a + [a ,x]T(\omega, y)a = 0$, for all $x,y,\omega \in R$. .... (3)

Setting $x=a$ in (3), we get:

$[T(a, y),a]\omega a = 0$, for all $y,\omega \in R$. .... (4)

Replacing ω by $\omega T(a, y)$ in (4) leads to:

$[T(a, y),a]\omega T(a, y)a = 0$, for all $y,\omega \in R$. ...... (5)

Also, right multiplication of (4) by $T(a, y)$ gives:

$[T(a, y),a]\, \omega a\, T(a, y) = 0$, for all $y,\omega \in R$. .... (6)

Subtracting (6) from (5), we arrive at:

$$[T(a, y),a]\, \omega[T(a, y),a]=0, \text{ for all } y,\omega \in R.$$

Using the semiprimeness of R, we have:

$[T(a, y),a]=0$, for all $y \in R$............................ (7)

Right multiplication of (7) by a gives:

$$[T(a, y),a]a=0, \text{ for all } y \in R.$$

In view of (1), the above relation can be reduces to:

$$a\, y\, a = 0, \text{ for all } y \in R.$$

The semiprimeness of R leads to $a=0$, hence φ is free action.■

Theorem (3.9):

Let R be a 2-torsion free semiprime ring and D: $R\times R\longrightarrow R$ be a Biderivation, then the mapping φ: $R\times R\longrightarrow R$ defined by $\varphi(x, y) = [D(x, y), x]$, for all $x,y \in R$ is free action.

Proof:

Let $a \in \mathcal{D}(\varphi)$, then $\varphi(x, y)a = ayx$, for all $x,y \in R$, that is:

$[D(x, y), x]a = ayx$, for all $x,y \in R$. .... (1)

Linearization of (1) with respect to x, we get:

$[D(x, y), \omega]a + [D(\omega, y), x]a = 0$, for all $x,y,\omega \in R$. .... (2)

Putting ωainstead of ω in (2), and using (2), we obtain:

$$\omega[D(x, y),a]a + D(\omega, y)[a ,x]a + [\omega, x]D(a, y)a + \omega [D(a, y),x]a=0, \text{ for all } x,y,\omega \in R.$$

Setting $x=a$ in the above relation leads to:

$$2\omega[D(a, y),a]a + [\omega, a]D(a, y)a = 0, \text{ for all } y,\omega \in R.$$

That is

$\omega[D(a, y),a]a + [\omega D(a, y),a]a = 0$, for all $y,\omega \in R$. .... (3)

Substituting a for ω in (3), then using the fact that R isa 2-torsion free ring, we arrive at:

$a [D(a, y),a]a = 0$, for all $y \in R$. .... (4)

In view of (1), the above relation reduces to:

$a^2y a = 0$, for all $y \in R$. .... (5)

Right multiplication of (5) by a, since R is a semiprime ringwe get first that $a^2=0$, and consequently $a=0$. Hence φ is free action. ■

Corollary (3.10):

Let R be a semiprime ring, and(T, S) be a double Bicentralizer. Then the mapping φ: $R\times R\longrightarrow R$ defined by $\varphi (x, y) = [T(x, y)-S(x, y), x]$, for all $x,y \in R$ is a free action.

Proof:

Using Lemma (2.11) and Theorem (3.9), we get the assertion of the corollary.

Theorem (3.11):

Let R be a semiprime ring, and (T, S) is double Bicentralizer of R. In this case $D(T) = D(S)$.

Proof:

For any $x,y,z \in R$, we have:

$zT(x, y) = S(y, z)x$ .... (1)

Let $a \in D(T)$, then by Theorem (3.1) we have $a \in Z(R)$ and

$T(x, y)a = ayx$, for all $x,y \in R$. .... (2)

Now, right multiplication of (1) by a leads to:

$zT(x, y)a = S(y, z)xa$, for all $x,y,z \in R$. .... (3)

According to (2), and since $a \in Z(R)$, the above relation reduces to:

$$zayx = S(y, z)ax, \text{ for all } x,y,z \in R.$$

That is

$$(S(y, z)a - azy)x = 0 , \text{ for all } x,y,z \in R.$$

The semiprimeness of R leads to:

$$S(y, z)a = azy, \text{ for all } y,z \in R.$$

Hence$a \in D(S)$. It follows that $D(T) \subseteq D(S)$.

Conversely, let $a \in D(T)$, then$a \in Z(R)$ by Theorem (3.1). Also

$S(y, z)xa = S(y, z)ax = azyx$, for all $x,y,z \in R$. .. .. (4)

In view of (4) and the fact that $a \in Z(R)$, the relation (3) reduces to:

$$zT(x, y)a = zayx, \text{ for all } x,y,z \in R.$$

Therefore

$$z (T(x, y)a - ayx)=0, \text{ for all } x,y,z \in R.$$

Thesemiprimeness of R leads to $T(x, y)a = ayx$, for all $x,y \in R$, that is$a \in D(T)$.

Hence $D(S) \subseteq D(T)$. ■

Theorem (3.12):

Let (T, S) be a double Bicentralizer of a semiprime ring R. Then the mapping φ: $R\times R\longrightarrow R$ defined by $\varphi(x, y) = T(x, y)x + S(x, y)x$, for all $x,y \in R$ is free action.

Proof:

Let $a \in D(\varphi)$, then we have:

$$\varphi(x, y) = ayx, \text{ for all } x,y \in R.$$

That is

$(T(x, y)x+S(x, y)x)a = ayx$, for all $x,y \in R$. .... (1)

By hypotheses

$zT(x, y)=S(y, z)x$, for all $x,y,z \in R$. .... (2)

As special case of (2) when $z = x$, we have:

$xT(x, y)=S(x, y)x$, for all $x,y \in R$. .... (3)

Now, according to (3), the relation (1) becomes:

$(T(x, y)x+xT(x, y))a = ayx$, for all $x,y \in R$. .... (4)

Linearization of (4) with respect to x leads to:

$(T(x, y)\omega + \omega T(x, y)+T(\omega, y)x+xT(\omega, y))a = 0$, for all $x,y, \omega \in R$. .... (5)

Setting$x=\omega=a$, the above relation reduces to:

$$2(T(a, y)a + aT(a, y))a = 0, \text{ for all } x,y, \omega \in R.$$

According to (4), the above relation reduces to:

$2a y a = 0$, all $y \in R$. .... (6)

The substitution xa for ω in (5) gives:
$(T(x, y)xa + xaT(x, y) + T(x, y)ax + xT(x, y)a)a = 0$, for all $x, y, \omega \in R$.

Consequently
$(T(x, y)x + xT(x, y))a^2 + T(x, y)axa + xaT(x, y)a = 0$, for all $x, y \in R$.
According to (4) and (6), the above relation reduces to:
$ayxa - T(x,y)\,axa + xaT(x, y)a = 0$, for all $x, y \in R$. .... (7)

Setting $x = a$ in (7), we obtain:
$aya^2 - T(a, y)a^3 + a^2T(a, y)a = 0$, for all $y \in R$. .... (8)

On the other hand, the substitution a for x in (4) leads to:
$T(a, y)a^2 + aT(a, y)a = aya$, for all $y \in R$. ...... (9)

Right multiplication of (9) by a, we get:
$T(a, y)a^3 + aT(a, y)a^2 = aya^2$, for all $y \in R$. .. (10)

According to (10) the relation (8) reduces to:
$aT(a, y)a^2 + a^2T(a, y)a = 0$, for all $y \in R$. ..... (11)

Combining the relations (9) and (11), we arrive at:
$a^2y\,a = 0$, all $y \in R$. .... (12)

Again, multiplying (12) from the right by a, then using the semiprimeness of R we arrive at $a^2 = 0$, thus $a = 0$, that is φ is free action. ■

Theorem (3.13):

Let R be a 2-torsion free prime ring with identity and $S, T: R\times R\longrightarrow R$ is a symmetric left and right Bicentralizer respectively. Suppose $c \in R$ is adependent element of the mapping φ: $R\times R\longrightarrow R$ defined by $\varphi(x, y) = bS(x, y) + T(x, y)b$, for all $x, y \in R$ and $b \in R$ be a fixed element. In this case $c^2 \in Z(R)$.

Proof:

According to Lemma (2.8), there exist a fixed element $a \in R$ such that $T(x, y) = ayx$, and $S(x, y) = xya$ for all $x, y \in R$.
We shall assume that $a \neq 0$ and $b \neq 0$, then we have:
$(bS(x, y) + T(x, y)b)c = cyx$, for all $x, y \in R$.

That is
$(bxya + ayxb)c = cyx$, for all $x, y \in R$. .... (1)
Putting $x\omega$ for x in (1), we get:
$bx\omega yac + ayx\omega bc = cyx\omega$, for all $x, y, \omega \in R$. .... (2)

Right multiplication of (1) by ω gives:
$bxyac\omega + ayxbc\omega = cyx\omega$, for all $x, y, \omega \in R$. .... (3)

Subtracting (3) from (2), we obtain:

$bx[\omega, yac] + ayx[\omega, bc] = 0$, for all $x, y, \omega \in R$. .... (4)

Now, setting $y=1$, then the relations (1) and (4) becomes:
$bxac + axbc = cx$, for all $x, \omega \in R$. .... (5)

$bx[\omega, ac] + ax\,[\omega, bc] = 0$, for all $x, \omega \in R$. .. (6)

Replacing x by cx in (6), we obtain:
$bcx\,[\omega, ac] + acx\,[\omega, bc] = 0$, for all $x, \omega \in R$. .... (7)

Left multiplication of (7) by ω gives:
$\omega bcx\,[\omega, ac] + \omega acx\,[\omega, bc] = 0$, for all $x, \omega \in R$. .... (8)

Putting ωx instead of x in (7), we get:
$bc\omega x\,[\omega, ac] + ac\omega x\,[\omega, bc] = 0$, for all $x, \omega \in R$. .... (9)

Subtracting (9) from (8), we arrive at:
$[\omega, bc]\;x\;[\omega, ac] + [ac, \omega]\;x\;[\omega, bc] = 0$, for all $x, \omega \in R$.
Using Lemma (2.1), we get:
$[\omega, ac]\;x\;[\omega, bc] = 0$, for all $x, \omega \in R$.
Suppose that $[\omega, bc] \neq 0$. This implies that $[\omega, ac] = 0$, for all $\omega \in R$, that is $ac \in Z(R)$.

Therefore the relation (6) reduces to:
$ax[\omega, bc] = 0$, for all $x, \omega \in R$.

Since $a \neq 0$, then $[\omega, bc] = 0$, for all $\omega \in R$, this means that $bc \in Z(R)$.

Now, we have both ac and bc are elements in $Z(R)$, then setting $x=c$ in (5), we obtain:
$2\,(ac)\,(bc) = c^2$, for all $x, \omega \in R$, hence $c^2 \in Z(R)$. ■

Corollary (3.14):

Let R be a 2-torsion free prime ring with an identity element and let $a, b \in R$ be a fixed elements. Suppose $c \in R$ is a dependent element of the mapping φ: $R\times R\longrightarrow R$ defined by $\varphi(x, y) = bxya + ayxb$, for all $x, y \in R$. In this case the following statement holds.

(1) $ac \in Z(R)$ and $bc \in Z(R)$.
(2) $(ba + ab)c = c$.
(3) $c^2 \in Z(R)$.

Proof:

According to Theorem (3.13), it follows that ac and $bc$are elements in $Z(R)$. Also, for a dependent element $c \in R$, we get $c^2 \in Z(R)$. So we have only to show that $(ba + ab)c = c$.
Now, from relation (5) of Theorem (3.13) we have:

$$(bxa + axb)c = cx \text{ , for all } x,y \in R.$$

Setting $x = c$ in the above relation we get the second statement of this corollary.■

Theorem (3.15):

Let R be a prime ring with an identity element and extended centroid C. Let S, T:$R\times R \longrightarrow R$ is a symmetric left and right Bicentralizer respectively, and $b \in R$ be a fixed element. Then the mapping φ: $R\times R \longrightarrow R$ defined by $\varphi(x, y) = T(x, y)b - bS(x, y)$, for all $x,y \in R$ is a free action.

Proof:

Again using Lemma (2.8), for a fixed element $a \in R$ we have $T(x, y)=ayx$ and $S(x, y)=xya$, for all $x,y \in R$. Let $c \in D(\varphi)$, then we have:

$(T(x, y)b - bS(x, y))c=cyx$,for all $x,y \in R$.

That is

$(ayxb - bxya)c=cyx$, for all $x,yv \in R$. .... (1)

We shall assume that $a\neq 0$ and $b\neq 0$, moreover we shall assume that a and b are C-independent, otherwise $\varphi(x, x)=0$, for all $x \in R$.
Replacing x by $x\omega$ in (1), we have:

$(ayx\omega b - bx\omega ya)c=cyx\omega$,for all$x,y, \omega \in R$. .... (2)

Right multiplication of (1) by ω gives:
$(ayxb - bxya)c\omega=cyx\omega$,for all$x,y, \omega, \in R$. .... (3)

Subtracting (3) from (2), we arrive at:
$ayx[\omega,bc] - bx[\omega,yac] =0$, for all$x,y, \omega \in R$. .... (4)

Now, Choosing$y$ the identity element, we can rewrite the relations (1) and (4) as:
$(axb - bxa)c=cx$,for all$x,y \in R$. .... (5)

$ax[\omega,bc] - bx[\omega,ac] =0$, for all $x , \omega \in R$. .... (6)

$acx[\omega,bc] - bcx [\omega,ac] =0$, for all $x,\omega \in R$. .... (7)

Left multiplication of (7) by ω gives:
$\omega acx\ [\omega, bc] - \omega bcx\ [\omega, ac] =0$, for all $x,\omega \in R$. .... (8)

Also, putting ωx instead of x in (7) leads to:
$ac\omega x\ [\omega, bc] - bc\omega x\ [\omega, ac] =0$, for all $x,\omega \in R$. .... (9)

Subtracting (9) from (8), we obtain:
$[\omega, ac]\ x\ [\omega, bc] - [\omega, bc]\ x\ [\omega, ac] =0, x, \omega \in R$. .... (10)

Suppose that $ac\notin Z(R)$. In this case there exist $\omega \in R$ such that $[\omega, ac]\neq 0$. It follows from relation (10) and Lemma (2.2) that there exist $\lambda_{\omega} \in C$ such that:
$[\omega, bc]= \lambda_{\omega}\ [\omega, ac]$ .... (11)

According to (11), the relation (6) reduces to:
$(a\ \lambda_{\omega} - b)x\ [\omega, ac] =0$, for all $x , \omega, \in R$. .... (12)

Since $[\omega, ac]\neq 0$, it follows from (9) that $b=a\ \lambda_{\omega}$, contrary to the assumption that a and b are C-independent. Therefore we have proved that $ac \in Z(R)$. Due to this fact, the relation (5) reduces to:

$a\ x\ [\omega, bc] =0$, for all $x, \omega \in R$.

Hence it follows (recall that $a\neq 0$) that $[\omega, bc] = 0$, for all $\omega \in R$ and consequently $bc \in Z(R)$. Since both ac and $bc$are elements in $Z(R)$, one can rewrite the relation (1) as:

$$axbc - acbx=cx, \text{ for all } x,y \in R.$$

Setting $x=c$ in the above relation, we obtain $c^2 =0$, it follows $c=0$. ■

Theorem (3.16):

Let R beaprime ring with an identity element and extended centroid C.Let T: $R\times R \longrightarrow R$be a symmetric left Bicentralizer. Suppose that $c \in R$ is dependent element of φ: $R\times R \longrightarrow R$ defined by $\varphi(x, y) = T(x, y)b$ for all $x,y \in R$. In this case $c=\lambda a$, where $0 \neq a,b$ be fixed elements and for some $\lambda \in C$.
Proof: Let $c \in \mathcal{D}(\varphi)$, then we have:

$$\varphi(x, y)c = cyx, \text{ for all } x,y \in R.$$

By Lemma (2.8) we have $T(x, y)=ayx$, for all $x,y \in R$ and a fixed element $a \in R$. So the above relation becomes:

$ayxbc = cyx$, for all $x,y \in R$............................ (1)

Putting xz instead of x in (1), we get:

$$ayxzbc = cyxz, \text{ for all } x,y,z \in R.$$

According to (1), we obtain $ayxzbc = ayxbcz$, that is

$$a\,y\,x[z, bc] = 0, \text{for all } x,y,z \in R.$$

Since R is prime ring and $a \neq 0$, we obtain:

$$x[z, bc] = 0, \text{for all } x,z \in R.$$

Consequently, $[z, bc] = 0$, for all $z \in R$, hence $bc \in Z(R)$, this makes it possible to rewrite relation (1) in the form:

$$Abcyx = cyx, \text{for all } x,y,z \in R.$$

That is

$$(abc - c)\,y\,x = 0, \text{for all } x,y \in R.$$

Using the primeness of R, we get:

$abc = c$ . ... (2)

Again, since $bc \in Z(R)$, the relation (1) can be given as:

$aybcx = cyx$, for all $x,y \in R$............................ (3)

Since R is a prime ring, the relation (3) reduces to:

$aybc = cy$, for all $y \in R$. .... (4)

Replacing y by ya in (4), we arrive because of (2) at:

$$ayc = cya, \text{ for all } y \in R.$$

Whence $c = \lambda a$by Lemma (2.2).■

Reference

[1] Ali F. and Anwar M., *"Dependent elements of Derivations on Semiprime Rings*", International Journal of Mathematics and Mathematical Sciences, Volume 2009, 1-6, (2009).

[2] Brešar M., "*Jordan derivationson Semiprime rings*", Proc. Amer. Math. Soc 104, no. 4, 1003-1006, (1988).

[3] Choda H., Kasahara I. and Nakamoto R., *"Dependent elements of automorphisms of C^*-algebra"*, Proc. Japan Acad. 48, 561-565, (1972).

[4] HeresteinI. N, "*Rings with involution*", The University of Chicago Press, Chicago (1976).

[5] Kallman R. R., "*A generalization of free action*", Duke Math. J. 36, 781-789, (1969).

[6] Kharchenko V. K., "*Automorphisms and derivations of associative rings*", Kluwer Academic Publishers, (1991).

[7] Laradji A. and Thaheem A. B., *"On Dependent elements in Semiprime rings*", Math. Japonica 47, 29-31, (1998).

[8] Molnar L., *"Jordan ∗-Derivation Pairs on A complex ∗-Algebra"*, Aequation Math. 54, 44-55, (1997).

[9] Murray F. J. and von NumannJ., *"On rings of operators*", Ann. Math. 37, 116-229, (1936).

[10] Muhammad A. and Mohammad S. S., "*Free action on Semiprime Rings*", Mathematica Bohemica, Vol. 133, No. 2, 197-208, (2008).

[11] Von Neumann J., *"On rings of operators III* ", Ann. Math. 41, 94-161, (1940).

[12] Vukman J. and Kosi-Ulbl I., "*On Dependent elements in rings*", International Journal of Mathematics and Mathematical Sciences 54, 2895-2906, (2004).

[13] Vukman J. and Kosi-Ulbl I., *"Centralizers on Ringsand algebra*", Bull. Austral. Math. Soc. 71, 225-234, (2005).

[14] Vukman J., *"On Dependent elements and related problems in rings*", Int. Math. J. 6, 93-112, (2005).

الخلاصة

قدمنا في هذا البحث مفهوم العناصر المعتمدة للدوال ثنائية الخطية ومفهوم الدوال ثنائية الخطية ذات التأثير الحر، واخترنا دوال ثنائية التمركز المعرفة على الحلقات شبه الأولية والدوال المتضمنة لها كنوع من هذه الدوال لتطبيق هذه المفاهيم ودراسة الصفات الخاصة لعناصرها المعتمدة وبيان الشروط التي تكون تحتها هذه الدوال حرة التأثير.