

## Some Notes on Taylors Series Expansion with ODE'S

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### Abstract

In this paper, our aim is to study the numerical solution of initial value problems (IVPs) for ODE's is one of the fundamental problems in scientific computation.

There are many well-established algorithms for approximate solution of IVPs. However, traditional integration methods usually provide only approximate values for the solution. Precise error bounds are rarely available the error estimates, which are sometimes delivered, are not guaranteed to be accurate and are sometimes unreliable. The main goal of the paper is to present all the existent approaches together emphasizing that since, all researches face the same problem but in different contexts, they are finding the same kinds of problems in spite of there different for mails m5 and methodologies.

Keywords: IVPs for ODE's, Taylor model methods, error bounds for taylor's series, approximation function by Taylor polynomials.

### 1-Introduction

Berz and his co-workers [1] have developed Taylor model methods, which combine interval arithmetic with symbolic computation [2, 5, 25, 27, 28]. In Taylor model methods the basic data type is not a single interval, but a Taylor model.

U:  $P_n(x) + i$

consisting of multivariate polynomial  $P_n(x)$  of order  $n$  in  $m$  variables, and a remainder interval  $i$ . In computations that involve  $U$ , the polynomial part is propagated by symbolic calculations wherever possible, and thus not significantly affected by the dependency problem of the wrapping effect, only the interval remainder term and Polynomial terms of order higher than  $n$ , which are usually small, are bounded using interval arithmetic.

In contrast reliable integration computes guaranteed bounds for the flow of an ODEs, including all discretization and roundoff errors on the computation originated by Moore in the 1960s [33], interval computations are a particularly useful tool for this purpose. There is a vast literature on interval methods for verified integration [2,5,25,27,28,33, 42], But there are still many open questions.

Taylor model arithmetic is an extension of interval arithmetic with a comprehensive variety of applicable enclosure sets. Nevertheless, there has been same debate

about the usefulness and the limitations of Taylor model methods [42]. To sometimes cursory description of technical details of Taylor model arithmetic, which may be obvious to the experts of Taylor models, but which are less trivial to others.

The motivation of this paper is to analyze Taylor model methods for the verified integration of ODEs and to compare these methods. Taylor models are better suited for integrating ODEs than interval methods whenever richness in available enclosure sets and reduction of the dependency problem is an advantage. Thus is usually the case for IVPs for nonlinear ODEs. Especially In combination with large initial sets or with large integration domains, although parameter intervals or initial sets can be handled by subdivision, this approach is only practical in low dimensions.

We use a simple nonlinear model problem to illustrate these advantages. The paper is structured as, follows. In the next section, basic concepts of interval arithmetic and Taylor model methods are reviewed interval methods for ODEs are represented. A nonlinear model problem is used to explain preconditioned Taylor model methods for ODEs in section 3. In the last section, numerical examples for linear ODEs are given.

**2-Dependency problem and wrapping Effect**

Interval methods are sometimes affected by overestimation, whence the computed error bounds may be overly pessimistic. Overestimation is often caused by the dependency problem, that is the failure of interval arithmetic to identify different occurrence of the same variable. For example the range of  $f(x):=x/(1+x)$  in (2,1) is  $[1/2, 2/3]$ , but interval – arithmetic evaluation yields

$$\frac{x}{1+x} = \left[ \frac{1,2}{2,3} \right] = \left[ \frac{1}{3}, 1 \right] \dots\dots\dots (2,1)$$

In general, the dependency problem is not easily removed. To diminish overestimation, alternative evaluation schemes, such as centered forms [33], have been development. A discussion of computer methods for the range of functions is given in [43].

A second source of overestimation is the wrapping effect, which appears when intermediate results of a computation are enclosed by intervals. The wrapping effect was first observed by Moore in 1965 [32]; a recent analysis has been given by Lohner [23].

**3- Taylor Model Arithmetic**

To reducing both the dependency problem and the wrapping effect, interval arithmetic has been extended with symbolic computations. Symbolic – numeric computations have been proposed under various names since the 1980s [11,16,23]. Early implementations in software were also given [11,15], but to the authors knowledge, these packages have not been widely distributed and are not available today.

Starting in the 1990s, Berz and his group developed a rigorous multivariate taylor arithmetic [2,25,28]. In these references, a taylor model is defined in the following way.

Let  $f: DC R^m \rightarrow R$  be a function that is (n+1) times continuously differentiable in an open set containing the box  $X$ . Let  $X_0$  be a point in  $X$ . Let  $P_n$  denote the  $n$ th order Taylor polynomial of around  $X_0$ , and let  $i$  be an interval such that

$$f(x) \in P_n(x - x_0) + i \text{ for all } x \in x \dots\dots\dots (3.1)$$

Then the pair  $(P_n, i)$  is called an  $n$ th order Taylor model of around  $X_0$  on  $X$ . This original definition of a Taylor model is useful for computations in exact arithmetic but it must be extended for floating point computations. For example, there is no Taylor model of  $e^x = 1 + x + (1/2) X^2 + (1/6) X^3 + \dots$  of order  $n \geq 3$  in IEEE 754 floating point arithmetic, since the coefficient of  $X^3$  is not exactly representable as a floating point number. In [29], Instead of the Taylor polynomial of  $f$ , an arbitrary polynomial  $P_n$  with floating point coefficients is used in (3.1), but the definition of Taylor model in [29] assume that the width of  $i$  is of order  $O(|w(x)|^n)$ . In this paper, such an assumption on the width of  $i$  is not required.

We use calligraphy letters for denoting Taylor models:

$$U: P_n(x) + i, X \in X$$

Where  $x \in IR^m, I \in IR$  are intervals and  $P_n$  is an  $m$ - variate polynomial of order  $n$ .  $x$  is called the domain interval of  $U$ , and  $i$  is its remainder interval. A Taylor model is the set of all  $m$ - variate continous functions  $f$  such that  $f(x) \in P_n(x) + i$

Holds for all  $X \in X$ . Evaluating  $U$  for all  $X \in X$ , we obtain the range of  $U$ :

$$R_2(U): = \{Z = P(X) + i \mid X \in X, i \in i\}$$

**Example (3.1):**

Taylor models of  $e^x$  and  $\text{Cos}x$ . Let  $x = [-\frac{1}{2}, \frac{1}{2}]$  and  $X_0 = 0$ . Then Taylor's theorem is a natural starting point for constructing Taylor models, we have

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots + \frac{1}{n!}x^n, \dots, |x| \leq 1, \text{cos}x \\ &= 1 - \frac{1}{2}X^2 + \frac{1}{24}X^4, \dots, \left(x = \frac{5\pi}{180}\right). \end{aligned}$$

From which we derive Taylor models for  $f_1(x) := e^X$  and  $f_2(x) := \text{cos}x$

$$U_{1(x)} := 1 + x + \frac{1}{2} x^2 + [-0.035, 0.035],$$

$$U_{2(x)} := 1 - \frac{1}{2} x^2 + [-0.010, 0.010], x \in x$$

**Respectively**

Taylor model arithmetic has been defines in [2,25,28]. We use the same arithmetic rules, even though our Taylor models differ slightly from the taylor models defined in these references. The difference only affects the function set that is defined by a Taylor model.

In computation that involve a Taylor model U, the polynomial part is propagated by simplic calculations wherever possible. In floating point computations, the round off errors of the symbolic operations are rigorously estimated and the estimate is added to the remainder interval of the final result. This part of computation is hardly affected by the dependency problem or the wrapping effect. Only the interval remainder term and polynomial terms of order higher than n (which in applications are usually small) are processed according to the rules of interval arithmetic.

**Example (3.2):**

Composition of two univariate Taylor models of order 2. Let  $X = [-\frac{1}{2}, \frac{1}{2}]$  and  $U_1(x) = 1 + x + \frac{1}{2}x^2 + [-0.035, 0.035]$ ,  $U_2(x) = 1 - \frac{1}{2}x^2 + [-0.010, 0.010]$  where  $x \in x$ .

It is tempting to compute the composition  $U_1 \circ U_2$  in the following manner

$$\begin{aligned}
 U_1(x) \circ U_2(x) &\subseteq 1 + \left[ 1 - \frac{1}{2}X^2 + [-0.010, 0.010] \right] + \frac{1}{2} \left[ 1 - \frac{1}{2}X^2 + [-0.010, 0.010] \right] + [-0.035, 0.035] \subseteq \\
 &\left[ 2 - \frac{1}{2}X^2 + [-0.045, 0.045] \right] + \\
 &\frac{1}{2} \left[ 1 - X^2 + \frac{1}{4}X^2 + [0.020, 0.020] - X^2 \right] + \\
 &\frac{1}{2} [-0.010, 0.010] + [-0.001, 0.001] \subseteq \frac{5}{2} - X^2 \\
 &\quad + \frac{1}{8}X^4 \\
 &-X^2 [-0.005, 0.005] + [-0.056, 0.056] \subseteq \frac{5}{2} \\
 &\quad - X^2 + [0, 0.008] \\
 &\quad - [-0.002, 0.002] + [-0.056, 0.056] \\
 &\quad = \frac{5}{2} - X^2 + [-0.058, 0.066]
 \end{aligned}$$

Hence, we may define

$$U_1(x) \circ U_2(x) = \frac{5}{2} - X^2 + [-0.058, 0.066] \dots \dots \dots (3.2)$$

However, the above computation does not yield a taylor model for  $e^{\cos x}$  for all  $x \in x$ . Evaluating (3.2) at  $X=0$ , we obtain

$$U_1(0) \circ U_2(0) = [2.442, 2.566] \not\subseteq e = e^{\cos x}$$

The reason for this failure lies in the range of  $U_2$ , which is not contained in  $x$ , composition of Taylor models are indeed computed above, but it is required that the domain of  $U_1$  contains the range of  $U_2$ .

In our example, it suffices to compute the remainder term for the exponential function on the interval  $[-1, 1]$ . Using Lagrange's representation of the remainder term, we have  $\frac{e}{3!}X^2 \in [-\frac{e}{6}, \frac{e}{6}] \subseteq [-0.45, 0.454]$  for all  $\xi \in [-1, 1]$  and all  $X \in [-1, 1]$ .

Using  $[-0.454, 0.454]$  instead of  $[-0.035, 0.035]$  in the derivation of (3.2) yields

$$U_1(x) \circ U_2(x) := \frac{5}{2} - X^2 + [-0.477, 0.485]$$

Which is a verified enclosure of  $U_1(x) \circ U_2(x)$  for all  $x \in x$ . Note that it is still not a verified enclosure for all  $x \in [-1, 1]$ . The latter requires that the interval term of  $U_2$  is also computed for is  $x \in [-1, 1]$ .

A Taylor model vector is a vector with Taylor model components. When no ambiguity arises we call a Taylor model vector simply a Taylor model. Arithmetic operations for Taylor model vectors are defined computation wise.

**4- Taylor Series in Several Variable**

For example, for a function that depends on two variables,  $x$  and  $y$  the taylor series to second order about the point  $(a, b)$  is

$$\begin{aligned}
 f(a, b) &= (x-a)f_x(a, b) + (y-b)f_y(a, b) + \frac{1}{2!} \\
 &[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)]
 \end{aligned}$$

Where the subscripts denote the respective partial derivatives.

**Example (4.1):**

Compute a second-order Taylor series expansion around

point (a,b)=(0,0) of a function

$$f(x,y)=e^x \log(1 + y).$$

Firstly, we compute all partial derivatives we need

$$f_x(a,b)= e^x \log(1 + y)|_{(x,y)=(0,0)} =0,$$

$$f_y(a,b)= \frac{e^x}{(1+y)}|_{(x,y)=(0,0)} =1,$$

$$f_{xx}(a,b)= e^x \log(1 + y)|_{(x,y)=(0,0)} =0,$$

$$f_{yy}(a,b)= -\frac{e^x}{(1+y)^2}|_{(x,y)=(0,0)} =-1,$$

$$f_{xy}(a,b)=f_{yx}(a,b)= \frac{e^x}{(1+y)}|_{(x,y)=(0,0)} =1.$$

The Taylor series is

$$T(x,y)=f(a,b)+(x-a)f_x(a,b)+(y-b)$$

$$f_y(a,b)+1\left[2\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b) f_{xy}(a,b)+\right.\right.$$

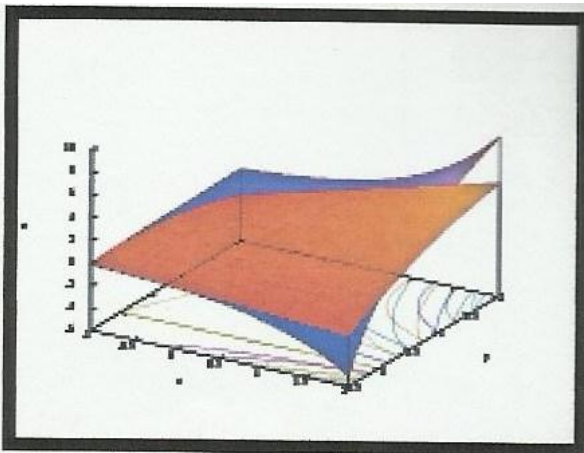
$$\left.\left.(y-b)^2 f_{yy}(a,b)\right]+\dots,$$

$$T(x,y)=0+0(x-0)+1(y-0)+1\left[2\left[(x-0)^2+2(x-0)(y-0)+(-1)(y-0)^2\right]+\dots\right.$$

$$\left. = y+xy-y^2\right]+2+\dots$$

Which in this case becomes

Since  $\log(1 + y)= y+xy-y^2\right]+2+\dots$  for  $|y|<1$ .



**Second –order Taylor approximation (in orange) of a function  $f(x,y)=e^x \log(1 + y)$  Around origin**

**5- Function approximation**

Suppose that we want to know the value of a function such as  $e^x$ . We can approximate the function  $e^x$  (which we can only evaluate on a calculator) and replace it by a polynomial which we can evaluate using pencil and paper

Before going on, we introduce one new term. The Taylor polynomial at a of degree n for a function f is the terms in the Taylor series up to degree n, thus this polynomial,  $p_n(x)$  is defined by

$$P_n(X) = \sum_{k=0}^n \frac{f^k(a)}{k!} (x - a)^k$$

Suppose we want to approximate  $e^x$ , or say e. We know from the previous section that  $e^x$  is given by the Taylor  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ . So that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

This series converges very rapidly and we shall see that we only need a few terms to find e to several decimal places of accuracy.

**Example (5.1):**

Compute e to within an error of at most  $10^{-3}$ .

**Solution.** According to Taylor's theorem with remainder, there is a number c between 0 and 1 so that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \leq \frac{e^c}{(N+1)!}$$

Take the largest and smallest values for  $e^c$  implies

$$\frac{1}{(N+1)!} e - \sum_{n=0}^N \frac{1}{n!} \leq \frac{e}{(N+1)!} \dots\dots\dots (5.1)$$

Thus we need to find N so that  $e/(N+1)! \leq 10^{-3}$ . We apparently have a problem. In order find N and compute e, we need to know the value of e. There are several ways out this circle. (1) Cheat-use the value of e from your calculator. This will be acceptable on sets. (2) Recall that when we defined e, we showed that  $e \leq 4$ . (3) In the example below, we will show how to use Taylor's theorem to find some Information about the size of e. We temporarily use (2) and thus the error is at most  $4/(N+1)!$  Consider the table below:

n	$4/(N+1)!$	$\sum_0^N \frac{1}{n!}$
4	$4/120 = 0.0333$	
5	$4/720 = 5.5 \cdot 10^{-3}$	
6	$4/5040 = 7.9 \cdot 10^{-4}$	2.71805...

We see from this table that N = 6 is the smallest value that will give allow us to approximate e with an error of at most  $10^{-3}$ . And if I entered the numbers correctly in my calculator the approximate value is 2.71805.

**Example (5.2):**

Use Taylor's theorem to get a rough bound on  $e$ .

**Solution.** We used the displayed equation (1) from the previous example with  $N=2$  and that  $e^c \leq e$  to obtain

$$\frac{1}{6} \leq e - \frac{5}{2} \leq e/6.$$

The left inequality gives

$$\frac{8}{3} \leq e.$$

While the right-hand one gives

$$\frac{5}{6} \leq e \frac{5}{2}$$

Which implies  $e \leq 3$ . Thus we conclude

$$\frac{8}{3} \leq e \leq 3$$

We consider an example for the cosine function.

**Example (5.3):**

Use Taylor's theorem to find an interval where

$$|\cos(x) - (1 - \frac{x^2}{2})| \leq 10^{-4}.$$

**Solution.** We recognize that  $1 - x^2/2$  is the Taylor polynomial of degree 2 for cosine at 0, or the McLaurin polynomial for cos.

From Taylor's theorem, we have that

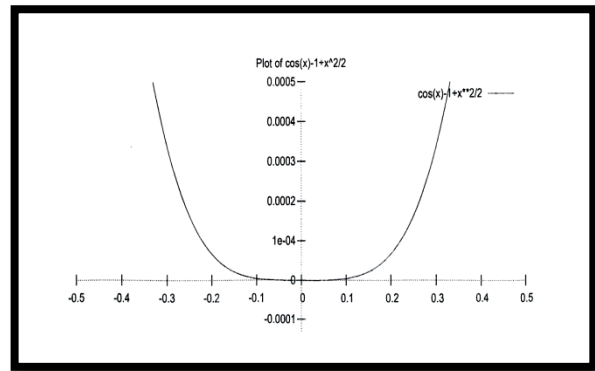
$$|\cos(x) - (1 - \frac{x^2}{2})| = |\sin(c) \frac{x^3}{3!}|$$

Since we know that  $|\sin(c)|$  is at most 1, the error will be at most  $10^{-4}$  if we have that

$$|\frac{x^3}{3!}| \leq 10^{-4}$$

Solving this inequality gives  $|x| < 0.084 \dots$

We can graph  $\cos(x) - 1 + x^2/2$  and see if we have done a good job.



Examining this graph shows that we have not done a good job. The error does not become larger than  $10^{-4}$  until  $x$  is about 0.2.

A moment's reflection will explain why we did not get the best possible answer. The second and third Taylor polynomials are equal because the terms of odd powers are 0. Thus, applying Taylor's theorem to estimate the difference between  $\cos(x)$  and its Taylor polynomial of degree three gives us

$$|\cos(x) - (1 - \frac{x^2}{2})| \leq \frac{x^4}{4!}.$$

Solving this inequality gives us

$$|x| \leq 0.22.$$

This agrees with the graph.

## 6- Taylor sequential applications

Taylor series for several benefits the most important because it allows the expression of any mathematical function through many border so we can find approximate solutions to the question s of whether the exact solution elusive. It also of great importance sequential Taylor in mathematics digital where many a algorithms to solve equations approved there on a sequential Taylor. It should be noted all the practical applications for the ending sequence which make it imperative that we take into account the precision which we want to get to in our solution to the equation. While the automatic Landing system bears a mistake between a meter or two meters at the landing site, the position of the head, which Likra data from the cylinder only accepts fault with a fraction of a millionth of a meter some familiar useful Taylor functions (series).

$$e^x, \log(1-x), \log(1+x), \frac{1-x^{m+1}}{1-x}, \frac{1}{1-x}, \frac{x^m}{1-x}, \text{etc.}$$

$$e^x \cos x, e^x \sin x, e^{\tan x}, e^{-x^2}$$

**7- Accuracy of Taylor series**

So how many terms should we use in getting a certain pre-determined accuracy in a Taylor series. One way is to use the formula for the Taylor theorem with remainder and its bounds to calculate the number of terms. This is shown in the example below.

**Example (7.1):**

So how many terms should we use in getting a certain pre-determined accuracy in a Taylor series one way is to use the formula for the Taylor's theorem remainder and its bounds to calculate the number of terms.

The Taylor series for  $e^x$  at point  $x=0$  is given by

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{4!}x^3 + \dots + \frac{1}{n!}x^n + R_n(x)$$

How many term it would require to get an approximation of  $e^1$  within magnitude of true errors of less than  $10^{-6}$ ? Using  $(n+1)$  terms of Taylor series given errors bounds of

$$R_n(x) = \frac{(x-h)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

For our example  
 $X=0, h=1, f(x) = e^x$

Hence  $R_n(0) = \frac{(0-1)^{n+1}}{(n+1)!} f^{(n+1)}(c) = \frac{(-1)^{n+1}}{(n+1)!} e^c$

Since  $x < c < x + h$   
 $0 < c < 0 + 1$   
 $0 < c < 1$

$\frac{1}{(n+1)!} < |R_n(0)| < \frac{e}{(n+1)!}$ , so if we want to find out how many terms it would require to get an approximation of  $e^1$  with a magnitude of true error of less than  $10^{-6}$

$$\frac{e}{(n+1)!} < 10^{-6} \Rightarrow (n+1)! > 10^6 e$$

$$\Rightarrow (n+1)! > 10^6 x3$$

as we do not know the value of  $e$  but it is less than 3.

So  $n \geq 9$  are needed to get  $e^1$  within an error of  $10^{-6}$  in value.

**Conclusion**

We have compared traditional enclosure methods with Taylor model based integration. For the verified solution of initial value problems for ODEs, we have shown how Taylor model methods benefit from symbolic computations. Increased flexibility in admissible boundary curves of enclosures is an intrinsic advantage over traditional interval methods, not only for the solution of ODEs. In future research, we hope to contribute to the further development and increased use of Taylor model methods.

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#### الخلاصة

في هذا البحث هدفنا هو دراسة الطول العددية لمسائل القيم الأولية (IVPs) إي من المعادلات التفاضلية الاعتيادية (ODEs) واحدة من المسائل الأساسية في الحسابات العلمية.

هناك عدد من الخوارزميات الرصينة للحلول التقريبية لمسائل القيم الأولية (IVPs), على هذا النحو طرق التكملات التقليدية عادة توفر قيم تقريبية للحلول. حدود الأخطاء الدقيقة نادرا ما توفر تخمين للخطأ , الذي أحيانا لا يسلم مضمونا ليكو دقيقا وفي أحيان أخرى لا يعول عليه.

الشئ الرئيسي للبحث الذي يحضر جميع القربيات الموجودة مع التأكيد بأنه بسبب جميع البحوث تواجه نفس المشكلة ولكن بسياقات مختلفة. وهي تواجه نفس نوع من المسائل بالرغم من اختلاف شكلياتها و منهجياتها.

كلمات مفتاحيه: القيم الأولية (IVPS) للمعادلات التفاضلية الاعتيادية (ODEs), طرق نماذج تيلر, حدود الأخطاء لتمسلسلات تيلر, تقريب الدوال مع متعددة تيلر.