Chebyshev Wavelets Method for Solving Partial Differential Equations of Fractional Order

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Abstract

In this paper, we use the second kind Chebyshev wavelet operational matrix of fractional integration for solving fractional order linear partial differential equations.

By using this method the fractional order partial differential equation is translated into Lyapunov type matrix equation and the computation effort become convenient. Two illustrative examples are provided to demonstrate the validity, simplicity and applicability of the numerical scheme based on the Chebyshev set of functions.

Keywords: Operational matrix, Chebyshev wavelets, Fractional order partial differential equations.

Introduction

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and wide spread fields of science and engineering. It does provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics, as well as, their extensions and generalizations in one and more variables, [1].

Fractional differential equations are generalized from classical integer-order ones, which are obtained by replacing integer- order derivatives by fractional ones.

Their advantages comparing with integer-order differential equations are the capability of simulating natural physical process and dynamic system more accurately, [3].

The partial differential equations involving derivatives with non-integer orders have shown to be adequate models for various physical phenomena in areas, such as damping laws, diffusion processes, etc. Other applications include electromagnetic, electrochemistry, arterial science, and the theory of ultra-slow processes and finance, [2].

However, several numbers of algorithms for solving fractional order partial differential equation have been investigated.

Suarez in [4] used the eigenvector expansion method to find the solution of motion containing fractional derivative. Podlubny [5] used the Laplace transform method to solve fractional differential equations numerically with Riemann-Liouville derivatives definition as well as the fractional partial differential equations with constant coefficients Meerscharet and Tadjeran [6] proposed the finite difference method to find the numerical solution of two-sided space-fractional partial differential equations. Momani [7] used a numerical algorithm to solve the fractional convection-diffusion equation with nonlinear source term. Odibat and Momani [11] used the variation iteration method to handle fractional partial differential equations in fluid mechanics. Jafari and Seifi [12] solved a system of nonlinear fractional partial differential equations using homotopy analysis method. Wu [2] derived a wavelet operational method to solve fractional partial differential equations numerically. Chen and Wu [8] used wavelet method to find the numerical solution for a class of fractional convection-diffusion equation with variable coefficients. Geng [9] suggested a wavelet method for nonlinear partial differential equations of fractional order.

Guo and et. al [10] used the fractional variationl homotopy perturbation iteration method to solve a fractional diffusion equation.

In this paper, the same approach given in [2] will be used but with the use of Chebyshev wavelets method to solve the fractional order partial differential equations. Wavelet analysis as a new approach of mathematics is widely applied in signal analysis, image manipulation, and numerical analysis, etc. It mainly studies the expression of functions, that is functions are decomposed into summation of "basic functions" and every "basic functions" is obtained by compression and translation of a mother wavelet function with good properties of locality and smoothness, which makes people can analyze the properties of locality and integer in process of expressing functions [13]. Beside their conventional applications in signal and image processing, wavelet basis had received attention dealing with numerical solutions of integer order as well as fractional order differential equations. Wavelet basis can be used to reduce the underlying problem to a system of algebraic equations by estimating the integrals using operational matrices [3], [15] and [16].

Recently the operational matrices of fractional order integration for the Haar wavelets, the Chebyshev wavelets and the Legendre wavelet have been developed in [14], [17], [18] and [19] to solve the fractional order differential equations.

2. Fractional Derivatives and Integration

There are several definitions of fractional derivatives of order $\alpha > 0$, ([20], [21]).

The two most commonly used definitions are the Riemann-Liouville and Caputo in which each definition uses Riemann-Liouville fractional integration and derivative of whole order. The difference between the two definitions is in the order of evaluation.

Rimann-Liouville fractional integration of order α is defined as:

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1} f(x)dx,\ \alpha > 0,\ t > 0$$

$$I^0 f(t) = f(t)$$

The next two equations define Riemann-Liouville and Caputo fractional derivatives of order α, respectively.

$$D^{\alpha}f(t) = \frac{d^n}{dt^n}\left(I^{n-\alpha}f(t)\right)$$

$${}^cD^{\alpha}f(t) = I^{n-\alpha}\left(\frac{d^n}{dt^n}f(t)\right)$$

where $n - 1 < \alpha \le n$ and $n \in IN$.

Properties of the fractional derivatives and integration can be found in [5], we mention the following:

For $f(t) \in C^m[a,b]$, $\alpha,\beta \ge 0$, $n-1 < \alpha \le n$, $\alpha + \beta \le m$, $v \ge -1$:

1- $(I^{\alpha}I^{\beta}y)(t) = (I^{\beta}I^{\alpha}y)(t) = (I^{\alpha+\beta}y)(t)$.

2- $(I^{\alpha}\, {}^cD^{\alpha}y)(t) = y(t) - \sum_{k=0}^{n-1} y^{(k)}(0^+)\frac{t^k}{k!}$.

3- $I^{\alpha}t^v = \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)}t^{\alpha+v}$.

3. The Second Kind Chebyshev Wavelet

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\psi(x)$ called the mother wavelet. When the dilation parameters a and the translation parameter b vary continuously we have the following family of continuous wavelet as [23], [24]:

$$\psi_{ab}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right),\ a, b \in ¡\ ,\ a \ne 0$$

If we restrict the parameter a and b to discrete values as $a = a_0^{-k}$ $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$, we have the following family of discrete wavelets:

$$\psi_{kn}(t) = |a_0|^{\frac{k}{2}}\psi(a_0^k t - nb_0),\ k, n \in ¢$$

where ψ_{kn} form a wavelet basis. In particular, when $a_0 = 0$ and then $\psi_{kn}(t)$ form an orthogonal basis.

The second kind Chebyshev wavelet $\psi_{nm}(t) = \psi(k,n,m,t)$ involve four arguments, $n = 1,2,...,2^{k-1}$, k is assumed to be any positive integer m is the degree of the second kind Chebyshev polynomials and t is the normalized time. They are defined on the interval [0,1) as [25]:

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}} \%_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t \le \frac{n}{2^{k-1}} \\ 0, & \text{otherwise} \end{cases}$$

.... (1)

where:

$$\tilde{U}_m(t)=\sqrt{\frac{2}{\pi}}\,U_m(t) \quad \text{.................................} (2)$$

and $m=0,1,...,M-1$. Here $U_m(t)$ are the second kind Chebyshev polynomials of degree m, with respect to the weight function $\omega(t)=\sqrt{1-t^2}$ on the interval $[-1,1]$ and satisfy the following recursive formula:

$$\begin{aligned}&U_0(t)=1,\quad U_1(t)=2t,\\&U_{m-1}(t)=2tU_m(t)-U_{m-1}(t),\ m=1,2,...\end{aligned} \quad ...(3)$$

We should note that in dealing with the second kind Chebyshev wavelet the weight function $\tilde{\omega}(t)=\omega(2t-1)$ have to be dilated and translated as:

$$\omega_n(t)=\omega(2^k t-2n+1) \quad \text{.............................} (4)$$

A function $f(x)\in L^2[0,1]$ may be expanded as

$$f(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(x) \quad \text{.........................} (5)$$

where:

$$\begin{aligned}c_{nm}&=\langle f(t),\psi_{nm}(t)\rangle_{\omega_n}\\&=\int_0^1\omega_n(t)\psi_{nm}(t)f(t)dt\end{aligned} \quad \text{..............................} (6)$$

in which < . , . > denotes the inner product in $L^2[0,1]$.

If the infinite series in Eq.(5) is truncated, then it can be written as:

$$f(t)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi(t)=C^T\Psi(t)=\tilde{f}(t) \quad ..... (7)$$

where C and $\Psi(t)$ are $2^{k-1}M\times 1$ matrices given by:

$$\begin{aligned}C=[&c_{1,0},c_{1,1},...,c_{1(M-1)},c_{2,0},c_{2,1},...,c_{2(M-1)},...,\\&c_{2^{k-1},0},c_{2^{k-2},1},...,c_{2^{k-1}(M-1)}]^T\end{aligned} \quad \text{....................} (8)$$

and

$$\begin{aligned}\Psi(t)=[&\psi_{1,0},\psi_{1,1},...,\psi_{1(M-1)},\psi_{2,0},\psi_{2,1},...,\\&\psi_{2(M-1)},...,\psi_{2^{k-1},0},\psi_{2^{k-2},1},...,\psi_{2^{k-1}(M-1)}]^T\end{aligned} \quad \text{...........} (9)$$

Taking the collocation points $t_i=\frac{2i-1}{2^k M}$.

We define the second kind Chebyshev wavelet matrix $\Phi_{m\times m}$ as:

$$\Phi_{m\times m}=\left[\Psi\left(\frac{1}{2m}\right),\Psi\left(\frac{3}{2m}\right),...,\Psi\left(\frac{2m-1}{2m}\right)\right] \quad \text{...............................} (10)$$

$$\tilde{f}_m=\left[\tilde{f}\left(\frac{1}{2m}\right)\tilde{f}\left(\frac{3}{2m}\right)...\tilde{f}\left(\frac{2m-1}{2m}\right)\right]=C^T\Phi_{m\times m} \quad \text{...............................} (11)$$

Correspondingly, we have:

Because the second kind Chebyshev wavelet matrix $\Phi_{m\times m}$ is an invertible matrix, the Chebyshev wavelet coefficient vector C^T can be gotten by:

$$C^T=\tilde{f}_m\Phi_{m\times m}^{-1} \quad \text{....................................} (12)$$

4. Operational Matrix of the Fractional Integration

The integration of the vector $\Psi(t)$ defined in Eq. (9) can be obtained as

$$\int_0^t\Psi(\tau)d\tau=P\Psi(t) \quad \text{...............................} (13)$$

where P is the m×m operational matrix for integration, [24].

Next, we shall present the derivation of the second kind Chebyshev wavelet operational matrix that was given in [25] and for this purpose we rewrite Riemann-Liouville fractional integration as following:

$$\begin{aligned}(I^\alpha f)(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-x)^{\alpha-1}f(x)\,dx\\&=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}*f(t)\end{aligned} \quad \text{....................................} (14)$$

where $\alpha\in\mathbb{R}$ is the order of the integration, $\Gamma(\alpha)$ is the Gamma function and $t^{\alpha-1}*f(t)$ denotes the convolution product of $t^{\alpha-1}$ and f(t). Now if f(t) is expanded in the second kind Chebyshev wavelet, the Riemann-Liouville fractional integration becomes

$$(I^{\alpha}f)(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t)$$

$$\approx C^T \frac{1}{\Gamma(\alpha)} \left\{ t^{\alpha-1} * \Psi(t) \right\} \quad \text{.......................... (15)}$$

Thus if $t^{\alpha-1} * f(t)$ can be integrated, then expanded in the second kind Chebyshev wavelets, the Riemann-Liouville fractional integration is solved via the second kind Chebyshev wavelet.

Also, we define an m-set of Block Pulse Function (BPF) as:

$$b_i(t) = \begin{cases} 1, & i/m \le t \le (i+1)/m \\ 0, & \text{otherwise} \end{cases} \quad \text{................ (16)}$$

Where $i = 0,1,\ldots,m-1$.

The functions $b_i(t)$ are disjoint and orthogonal. That is:

$$b_i(t)b_l(t) = \begin{cases} 0, & i \ne l \\ b_i(t), & i = l \end{cases} \quad \text{.......................... (17)}$$

The second kind Chebyshev wavelet may be expanded into an m-term Block Pulse Functions as:

$$\Psi(t) = \Phi_{m\times m} B_m(t) \quad \text{................................ (18)}$$

In Ref. [15] Kilicman and Al Zhour have given the Block Pulse operational matrix of the fractional integration F^{α} as following:

$$(I^{\alpha}B_m)(t) \approx F^{\alpha} B_m(t) \quad \text{.............................. (19)}$$

where:

$$F^{\alpha} = \frac{1}{m^{\alpha}} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_{m-1} \\ 0 & 1 & \zeta_1 & \zeta_2 & \cdots & \zeta_{m-2} \\ 0 & 0 & 1 & \zeta_1 & \cdots & \zeta_{m-3} \\ \vdots & \vdots & \ddots & 1 & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \zeta_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

with $\zeta_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

Next, we derive the second kind Chebyshev wavelet operational matrix of the fractional integration. Let

$$(I^{\alpha}\Psi)(t) \approx P^{\alpha}_{m\times m}\psi(t)$$

where matrix $P^{\alpha}_{m\times m}$ is called the second kind Chebyshev wavelet operational matrix of the fractional integration.

Using Eqs. (18) and (19), we have:

$$(I^{\alpha}\psi)(t) \approx (I^{\alpha}\Phi_{m\times m}B_m)(t)$$

$$= \Phi_{m\times m}(I^{\alpha}B_m)(t)$$

$$\approx \Phi_{m\times m}F^{\alpha}B_m(t) \quad \text{...................................... (21)}$$

From Eqs. (20) and (21), we get:

$$P^{\alpha}_{m\times m}\Psi(t) = P^{\alpha}_{m\times m}\Phi_{m\times m}B_m(t)$$

$$= \Phi_{m\times m}F^{\alpha}B_m(t) \quad \text{..................... (22)}$$

Then, the second kind Chebyshev wavelet operational matrix of the fractional integration P^{α} is given by:

$$P^{\alpha}_{m\times m} = \Phi_{m\times m}F^{\alpha}\Phi^{-1}_{m\times m} \quad \text{............................ (23)}$$

5. The Approach

In this section we shall use the numerical approach given by [2] to find the numerical solution for the linear fractional partial differential equations but by the second kind Chebyshev wavelet the integration of $Y(x,t) = \Phi^T(x)C\Phi(t)$ with respect to variable t yields:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} Y = \frac{\partial^{\alpha}}{\partial t^{\alpha}} \left(\Phi^T(x)C\Phi(t) \right)$$

$$= \Phi^T(x)C\left[\frac{\partial^{\alpha}}{\partial t^{\alpha}} \Phi(t) \right]$$

$$= \Phi^T(x)CP^{-\alpha}\Phi(t) \quad \text{........................ (24)}$$

Or

$$\frac{\partial^{-\alpha}}{\partial t^{-\alpha}} Y = \Phi^T CP^{\alpha}\Phi \quad \text{................................ (25)}$$

Similarly, the fractional integration order β of Y(x,t) with respect to variable x can be expressed as:

$$\frac{\partial^{-\beta}}{\partial x^{-\beta}} Y = \frac{\partial^{-\beta}}{\partial x^{-\beta}}\left(\Phi^T(x) C \Phi(t)\right)$$
$$= \left[\frac{\partial^{-\beta}}{\partial x^{-\beta}}\Phi(x)\right]^T C\Phi(t)$$
$$= \left[P^{\beta}\Phi(x)\right]^T C\Phi(t)$$
$$= \Phi^T(x)(P^{\beta})^T C\Phi(t)$$
$$= \Phi^T (P^{\beta})^T C\Phi \quad \text{..................................} (26)$$

In general, performing the double integration to the function Y(x,t) with fractional order α to variable t and fractional order β to variable x, we obtain:

$$\frac{\partial^{-\alpha}}{\partial t^{-\alpha}}\frac{\partial^{-\beta}}{\partial x^{-\beta}} Y = \Phi^T (P^{\beta})^T C P^{\alpha}\Phi \quad \text{................} (27)$$

Equations (25) and (26) are the main formulae for solving a fractional partial differential equation numerically via the second kind Chebyshev wavelet operational method.

The above procedure will be clear and illustrated by the following numerical examples given in the next section

6. Numerical Examples

In this section we will use the Chebyshev wavelet operational matrices of the fractional integration to solve linear fractional order partial differential equations and the results obtained using this scheme will be compare with the analytical solution or the solution obtained using other method or approaches.

Example 1:

Solve the following partial differential equation:

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 1, \quad x, t \ge 0 \quad \text{................................} (28)$$

with the initial conditions y(0,t) = y(x,0) = 0.

First we integrate the above equation with respect to t, yields to:

$$\int_0^t \frac{\partial y}{\partial x} dt + \left(y(x,t) - y(x,0)\right) = \int_0^t dt \quad \text{...............} (29)$$

then integrate (29) with respect to x, we obtain:

$$\int_0^t\int_0^x \frac{\partial y}{\partial x} dx dt + \int_0^x y\, dx = \int_0^t\int_0^x dx dt \quad \text{.....................} (30)$$

or:

$$\int_0^t \left(y(x,t) - y(0,t)\right) dt + \int_0^x y\, dx = \int_0^t\int_0^x dx dt \quad \text{.....} (31)$$

$$\int_0^t y\, dt + \int_0^x y\, dx = \int_0^x\int_0^t dt dx \quad \text{.............................} (32)$$

For solving the partial differential equation in (28) by the proposed method, we let $Y(x,t) = \Phi^T C \Phi$ and substitute (25) and (26) into (32), it gives:

$$\Phi^T C P \Phi + \Phi^T P^T C \Phi = \Phi^T P^T J P \Phi \quad \text{.......} (33)$$

where J is the matrix

$$J = (\Phi^T)^{-1}\begin{bmatrix} 1 & 1 & L & 1 \\ 1 & 1 & L & 1 \\ M & M & M & M \\ 1 & 1 & L & 1 \end{bmatrix}\Phi^{-1}$$

By multiplying $(\Phi^T)^{-1}$ to the left side and Φ^{-1} to the right side of each term in (33), it yields:

$$CP + P^T C = P^T J P$$

If m = 4 (k = 1, M = 2), solving the above equation yields:

$$C = \begin{bmatrix} 1.3125 & -0.5 & -0.2652 & -0.0884 \\ -0.5 & 0.3125 & 0.0884 & 0.0884 \\ -0.2652 & 0.0884 & 0.0625 & 0 \\ -0.0884 & 0.0884 & 0 & 0.0625 \end{bmatrix}$$

The matrix form of Y(x,t) is given by:

$$Y(x,t) = \Phi^T C \Phi$$

$$Y_{Ch} = \begin{bmatrix} 0.0625 & 0.125 & 0.125 & 0.125 \\ 0.125 & 0.3125 & 0.375 & 0.375 \\ 0.125 & 0.375 & 0.5625 & 0.625 \\ 0.125 & 0.375 & 0.625 & 0.8125 \end{bmatrix}$$

And the exact solution of the partial differential equation is given by:

$$y(x,t) = \begin{cases} t, & x \ge t \\ x, & t < x \end{cases}$$

Hence the matrix form of the exact solution is given by:

$$y_{exact}(x,t)=\begin{bmatrix} 0.125 & 0.125 & 0.125 & 0.125 \\ 0.125 & 0.375 & 0.375 & 0.375 \\ 0.125 & 0.375 & 0.625 & 0.625 \\ 0.125 & 0.374 & 0.625 & 0.8375 \end{bmatrix}$$

Example 2:

We use the operational method to solve the below fractional partial differential equation:

$$\frac{\partial^{\frac{1}{2}} y}{\partial^{\frac{1}{2}} x}+\frac{\partial^{\frac{1}{2}} y}{\partial^{\frac{1}{2}} t}=1, \ \ x,\ t \geq 0$$

with zero initial conditions

In case of m = 8 (k = 3, M = 2), we get the numerical solution by solving the Lyapunov equation:

$$(P^{1/2})^T C + C P^{1/2} = (P^{1/2})^T J P^{1/2}$$

Then yield:

$$C=\begin{bmatrix} 0.0392 & 0.0153 & 0.0528 & 0.0029 & 0.0569 & 0.0015 & 0.0593 & 0.0009 \\ 0.0153 & 0.0090 & 0.0249 & 0.0023 & 0.0283 & 0.0012 & 0.0303 & 0.0008 \\ 0.0528 & 0.0249 & 0.0790 & 0.0066 & 0.0888 & 0.0036 & 0.0947 & 0.0024 \\ 0.0029 & 0.0023 & 0.0066 & 0.0014 & 0.0089 & 0.0009 & 0.0104 & 0.0006 \\ 0.0569 & 0.0283 & 0.0888 & 0.0089 & 0.1024 & 0.0051 & 0.1108 & 0.0035 \\ 0.0015 & 0.0012 & 0.0036 & 0.0009 & 0.0051 & 0.0006 & 0.0062 & 0.0005 \\ 0.0593 & 0.0303 & 0.0947 & 0.0104 & 0.1108 & 0.0062 & 0.1213 & 0.0043 \\ 0.0009 & 0.0008 & 0.0024 & 0.0006 & 0.0035 & 0.0005 & 0.0043 & 0.0004 \end{bmatrix}$$

And the numerical solution of the above example using Chebyshev wavelet operational matrix will be similar to the Haar wavelet matrix given by [2], as below:

$$Y_{Chebyshev}=\begin{bmatrix} 0.1330 & 0.1881 & 0.2011 & 0.2098 & 0.2157 & 0.2201 & 0.2235 & 0.2262 \\ 0.1881 & 0.2888 & 0.3221 & 0.3425 & 0.3568 & 0.3675 & 0.3759 & 0.3827 \\ 0.2011 & 0.3221 & 0.3702 & 0.4003 & 0.4217 & 0.4379 & 0.4508 & 0.4614 \\ 0.2098 & 0.3425 & 0.4003 & 0.4376 & 0.4645 & 0.4853 & 0.5019 & 0.5157 \\ 0.2157 & 0.3568 & 0.4217 & 0.4645 & 0.4960 & 0.5205 & 0.5404 & 0.5569 \\ 0.2201 & 0.3675 & 0.4379 & 0.4853 & 0.5205 & 0.5482 & 0.5709 & 0.5899 \\ 0.2235 & 0.3759 & 0.4508 & 0.5019 & 0.5404 & 0.5709 & 0.5960 & 0.6171 \\ 0.2262 & 0.3827 & 0.4614 & 0.5157 & 0.5569 & 0.5899 & 0.6171 & 0.6401 \end{bmatrix}$$

Conclusions

In this paper, the numerical approach given by [2] was applied to solve fractional order partial differential equations but with the aid of the second kind Chebyshev wavelets operational matrices.

The results obtained of the tested examples are coincides with the results of [2] using the Haar wavelets.

The solution is convergent even though the size of increment may be large.

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الخلاصة

في هذا البحث، استحدمنا مصفوفه العمليات الناتجه من التكامل الكسري لمويجات تشيبيشيف لغرض حل المعادلات التفاضليه الجزئيه ذات الرتب الكسريه. بأستحدامنا لهذه الطريقه فأن المعادلات التفاضليه الجزئيه ذات الرتب الكسريه ستتحول الى مصفوفه معادلات من نوع ليابانوف والحسابات ستكون مناسبه اكثر. زودنا البحث بمثالين لتوضيح صلاحيه وبساطه وقابليه تطبيق خطوات الحل العددي المعتمد على مجموعه دوال تشيبيشيف.