# Certain Types of K-Spaces

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**Abstract**

In this work, we introduce and study several types of K-spaces and several properties of these types are proved. Among the obtained results are the following:

**1-** Let X be any space and Y be a K-space, then every compact map f: X $\longrightarrow$ Y is a closed map.

**2-** Let X be any space and Y be an sK-space, then every compact map f: X $\longrightarrow$ Y is s-closed map.

## 1. Introduction

In 1979, S. Kasahara [6] introduced the concept of operator associated with a topology $\Gamma$ of a space X as a map T from $\Gamma$ to P(X), and introduced the concept of T-compact space as a subset A of a topological space (X,$\Gamma$) is T-compact if for every open cover $\Omega$ of A, there exist a finite subcollection $\{C_1,C_2,\ldots,C_n\}$ of $\Omega$, such that $A = \bigcup_{i=1}^{n} T(C_i)$.

Following the Kasahara's ideas, in 1991 H. Ogata [7] introduced the concept of T-open sets and the concept of (T,L) continuous function. Also, H. Ogata introduces the notion of T-$T_i$ space, which generalize to $T_i$-space for i = 0, $\frac{1}{2}$, 1, 2; and studied some topological properties of such space.

In 1998, E. Rosas, J. Vielma [8] contained study by used S. Kasahara [6] definitions in some what modified form and his result to prove properties similar to the usual ones in general topology. And in 1999, modified the definition by allowing the operator T to be defined in P(X) as a map T from P(X) to P(X), such that $U \subseteq T(U)$, for every $U \in \Gamma$. Several researcher papers published in recent years using T-operator due to Ogata [7].

## 2. Preliminaries

In this section, some fundamental and basic concepts related with this paper are presented for completeness purpose:

***Definition (2.1), [2]:***

A space Y is said to be K-space, if:

1. Y is a $T_1$-space.
2. Every compactly closed subset of Y is closed.

In this work, we will introduce and study several types of K-spaces, where we mean by cl(A) and int(A) the closure and the interior of a subset A of a topological space X, respectively.

***Definition (2.2), [2]:***

Let Y be a topological space, $W \subseteq Y$, then W is called compactly closed if $W \cap K$ is compact for every compact set K in Y.

Notice that if W is closed in Y, then W is compactly closed. The converse is not necessarily true and this motivates the following definition:

***Definition (2.3), [3]:***

Let Y be a $T_2$-space, then Y is called a K-space if every compactly closed set is closed.

***Definition (2.4), [4]:***

Let X be apace, let $A \subseteq X$, then:

1. A is Semi-open if $A \subseteq cl(int(A))$.
2. A is Per-open if $A \subseteq int(cl(A))$.
3. A is $\alpha$-open if $A \subseteq int(cl(int(A)))$.

The compliment of semi-open (pre-open, $\alpha$-open) set is called semi-closed (pre-closed, $\alpha$-closed) set.

***Definition (2.5), [4]:***

Let f: X $\longrightarrow$ Y be a function then f is closed (semi-closed, pre-closed, $\alpha$-closed) if the image of every closed set in X is closed (semi-closed, pre-closed, $\alpha$-closed).

***Definition (2.6), [1]:***

f: X $\longrightarrow$ Y is called a proper map if f is continuous, closed and $f^{-1}(y)$ is compact in X for all $y \in Y$.

***Definition (2.7), [3]:***

f: X ⟶ Y is called a compact map if f is continuous and the inverse of each compact set K in Y is compact in X.

## 3. Main Results

We begin, by proving that the property of begin a K-space is a topological property.

***Theorem (3.1):***

Let X be a K-space and let f: X ⟶ Y be a homeomorphism, then Y is also a K-space.

***Proof:***

It is clear that Y is a $T_2$ – space

Now, suppose that W ⊆ Y is compactly closed

Hence, W ∩ K is compact for each compact set K in Y

Now, $f^{-1}$(W ∩ K) is compact in X, but:

$f^{-1}(W \cap K) = f^{-1}(W) \cap f^{-1}(K)$

This means that, $f^{-1}$(W) is a compactly closed in X, but X is a K-space, Hence $f^{-1}$(W) is a closed in X

Therefore, $f(f^{-1}(W)) = W$ is closed in Y this proves that Y is K-space. ■

***Theorem (3.2):***

Let X be a space, and A ⊆ X then A is compactly closed in X if f is the inclusion map i: A ⟶ X is a compact map.

***Proof:***

(⇒) Suppose A is compactly closed in X and let K be a compact set in X

Now, $i^{-1}(K) = A \cap K$, so A ∩ K is compact which means that i: A ⟶ X is a compact map.

(⇐) Assume that i: A ⟶ X is compact map, let K ⊆ X be compact hence $i^{-1}$(K) is compact but $i^{-1}(K) = A \cap K$, so A ∩ K is compact, ∀ K. This means that A is compactly closed. ■

***Theorem (3.3):***

Let X be space and Y be a K-space, then every compact map f: X ⟶Y is a closed map.

***Proof:***

Suppose W is a closed set in X and K be compact in Y

Now, $f^{-1}$(K) is compact in X

Hence, W ∩ $f^{-1}$(K) is compact

Now, $f(W \cap f^{-1}(K)) = f(W) \cap K$ is compact in Y

This implies that, f (W) is compactly closed, but Y is a K-space, hence f(W) is closed in Y

Hence, f is a closed map. ■

***Corollary (3.4):***

Every compact map f from any space X into a K-space Y is a proper map.

Now, we introduce the flowing definition:

***Definition (3.5):***

1. X is an sK-space if every compactly closed set in X is semi-closed (s-closed).
2. X is a pK-space if every compactly closed set in X is pre-closed (α-closed).
3. X is an αK-space if every compactly closed set in X is α-closed.

***Remark (3.6):***

It is known that if A ⊆ X is an α-open set, then A is semi-open and A ⊆ X is α-open if and only if A is semi-open and pre open. Hence X is an αK-space if f X is sK-space and pK-space.

***Theorem (3.7):***

Let X be any space and let Y be sK-space (pK-space, αK-space) then every compact map f: X ⟶Y is s-closed (p-closed, α-closed)

***Proof:***

Let W ⊆ X be closed set

Let K ⊆ Y be compact set

Now, $f^{-1}$(K) is compact and W ∩ $f^{-1}$(K) is compact

But, $f(W \cap f^{-1}(K)) = f(W) \cap K$ is compact, for each compact K this means that f(W) is compactly closed in Y

But Y is sK-space

Hence, f(W) is semi-closed this means that f: X ⟶ Y is s-closed

Similarly, if Y is pK-space (αK-space), then f: X ⟶ Y is p-closed (α-closed). ■

Now, we introduce the following definition:

***Definition (2.8):***

Let f:X⟶ Y be a continuous function, then f is said to be:

i) s-proper, if:

1. f is s-closed.
2. $f^{-1}(y)$ is compact, $\forall$ y $\in$ Y.

ii) p-proper, if:

1. f is p-closed.
2. $f^{-1}(y)$ is compact, $\forall$ y $\in$ Y.

iii) $\alpha$-proper, if:

1. f is $\alpha$-closed.
2. $f^{-1}(y)$ is compact, $\forall$ y $\in$ Y.

Now, the following corollary to theorem (3.7) may be stated:

***Corollary (3.9):***

Let X be any space and Y be an sK-space (pK-space, $\alpha$K-space). Then every compact map f: X ⟶Y is s-proper (p-proper, $\alpha$-proper).

**4. T*K-Space**

In this section, we introduce the concept of T* K-space first, but first we recall the following definition:

***Definition (4.1), [5]:***

Let (X, $\tau$) be a topological space and let T: p(X) ⟶ p(X) be a function such that W $\subseteq$ T (W), for each open set W in X. Then T is called an operator associated with the topology $\tau$ on X and the triple (X,$\tau$,T) is called operator topological space.

It is remarkable that if A $\subseteq$ X (which is not necessarily open in X satisfies A $\subseteq$ T(A), then we say that A is T*-open and the complement of T*-open is called T*-closed.

***Example (4.2):***

Consider ($\mathbb{R}$ ,$\tau_u$), where $\mathbb{R}$ is the set of real numbers and $\tau_u$ is the usual topology on $\mathbb{R}$ .

Define T: p($\mathbb{R}$ ) ⟶ p($\mathbb{R}$ ), as follows:

T(A) = cl (int (A)), A $\subseteq$ $\mathbb{R}$

then T is an operator associated with $\tau_u$

Let B = [0,1)

Then B is not open in $\tau_u$ but B satisfies B $\subseteq$ (B)

So B is T*-open (semi-open).

***Definition (4.3):***

Let (X,$\tau$,T) be an operator topological space, then X is called T*K-space if every compactly closed set in X is T*-closed.

***Theorem (4.4):***

Let X be any space and let (Y,$\tau$,L) be L*K-space (T is a topology on Y, and L is an operator associated with $\tau$), then every compact map f: X ⟶ (Y,$\tau$,L) is a L*-closed (that is f(W) is L*-closed in Y for each W closed in X).

***Proof:***

Let W $\subseteq$ X be closed let K $\subseteq$ Y be compact

Now $f^{-1}(K)$ is compact in X since W is closed, then W $\cap$ $f^{-1}(K)$ is compact

Also, f (W $\cap$ $f^{-1}(K)$) = f (W) $\cap$ K is compact in Y

This means that f(W) is compactly closed in Y

But Y is L*K-space

Hence f(W) is L*-closed

This means that f: X ⟶ (Y,$\tau$,L) is a L*-closed. ■

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