Reliable Algorithm for Solving Integro-Differential Equations of Fractional Order Using Homotopy Analysis Method

Fajer A. Abdul-Khaleq
Department of Computer Science, Dijlah University College, Baghdad-Iraq.

Abstract
In this paper, a powerful algorithm of the homotopy analysis method were developed and used to find theapproximate solution of integro-differential equations of fractional order. Which consist of fractional order ofdifferentiation and fractional order of integration.

The fractional derivative is described in the Caputo sense and the fractional integration is described in the Riemann-Liouville sense. In addition some examples are used to illustrate the accuracy and validity of this approach.

Keywords: Fractional Integro-Differential Equations, Homotopy Analysis Method.

Introduction

Fractional calculus has found diverse applications in various scientific and technological fields [1, 2, 3, 4], such as thermal engineering, acoustic, electromagnetism, control, robotics, viscoelasticity, diffusion, signal processing and many other physical processes.

There are several definitions of a fractional derivative of order $\alpha > 0$ [4], two most commonly used definitions are Riemann-Liouville and Caputo. Each definitions uses Riemann-Liouville fractional integration and derivatives of whole orders.

Integro-differential equations of fractional order may be considered as a branch of fractional integral equations which arise in modeling processing in applied sciences (physics, engineering, finance, biology, …), [5].

Throughout this paper we will exhibit integro-differential equations of fractional order of the form

$$D_*^\alpha y(t) = f(t) + I^\beta K[y(t)] \dots\dots\dots\dots\dots (1)$$

Where the fractional derivative is considered in Caputo sense of order α while the fractional integral is considered in Riemann-Liouville sense of order β. Integro-differential equations of fractional order are attacked by many researchers such as [5, 6, 7] where the fractional appeared in the derivative only. While [8, 9] treated the fractional order integro-differential equations in which the derivative and the integral are of fractional order using Adomian decomposition method and variational iteration method, respectively.

Another approach will be used in this paper to find the approximate solution of the integro-differential equations (1) using the reliable algorithm of the homotopy analysis method.

The homotopy analysis method (HAM) was first proposed by Lioa in his Ph.D. thesis [10]. A systematic and clear exposition on HAM is given in [11]. In recent years, this method has been successfully employed to solve many types of nonlinear, homogenous or non homogeneous equations and systems of equations, as well as, problems in science and engineering [10, 11, 12, 13, 14].

The HAM contains a certain auxiliary parameter to which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called $h$curve, it is easy to determine the valid region of h to gain a convergent series solution. Thus, through HAM, explicit analytic solutions of nonlinear problems are possible.

Odibat and etal [15] extended the application of the homotopy analysis method proposed by Lioa [10] to solve nonlinear differential equations of fractional order.

In this paper we shall use the approach of Odibat, [15] to find the approximate solution of integro-differential equations of fractional order given by equation (1).

Basic concepts [4]

In this section some definitions and properties related to fractional differentiation and integration are given:

the Riemann-Liouville fractional integration of order β is defined as,

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (x-s)^{\beta-1} f(s) ds, \beta \in \mathbb{R}^+, t > 0. \qquad (2)$$

The next two equations define Riemann-Liouville and Caputo fractional derivatives of order α, respectively,

$$D^\alpha f(t) = \frac{d^m}{dx^m}(I^{M-\alpha} f(t)). \qquad (3)$$

$$D_*^\alpha f(t) = I^{M-\alpha}(\frac{d^m}{dx^m} f(t)) \qquad (4)$$

Where $m-1 < \alpha \le m$ and $m \in N$.from now, Caputo fractional derivative will be denoted by D_*^α to maintain a clear distinction with Riemann-Liouville fractional derivative.

In the following we shall list some useful properties of Caputo fractional derivative.

1- Caputo introduced an alternative, definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

2- $I^\alpha D_*^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}$

Caputo fractional differentiation is a linear operator which is similar to integer order differentiation

$$D_*^\alpha[\lambda_1 f(t) + \lambda_2 g(t)]$$
$$= \lambda_1 D_*^\alpha f(x) + \lambda_2 D_*^\alpha g(x), \lambda_1, \lambda_2$$
$$\in \mathbb{R}$$

The Homotopy Analysis method [11]:

Consider the nonlinear equation in operatorform:

$$N[y(t)] = 0, t \in \mathbb{R} \qquad (5)$$

Where

$N=$ nonlinear operator

$y=$ unknown function

$t=$ the independent variable.

Let y_0 denote an initial gusess for the exact solution y, $h \ne 0$ an auxiliary parameter, $H(t) \ne 0 \ \forall t \in \mathbb{R}$ an auxiliary function and L an auxiliary linear operator with theproperty that $L[y(t)] = 0$ when $y(t)=0$. Then using q ∈

[0,1]as an embedding parameter, we construct such a homotopy which is called the zero order deformation:.

$$(1-q)L[\varphi(t,q) - y_0(t)] = qhH(t)N[\varphi(t,q)]. \qquad (6)$$

It should be emphasized that we have a great freedom to choose the initial guess y_0, the auxiliary linear operator L, the nonzero auxiliary function H(t). when q=0, the zero-order deformation equation (6) becomes

$$\varphi(t, 0) = y_0(t). \qquad (7)$$

and when q = 1, since h ≠ 0 and H(t) ≠ 0, the zero-order deformation equation (6) is equivalent to

$$\varphi(t, 1) = y(t) \qquad (8)$$

Thus, according to (7) and (8), as the embedding parameter q increases from 0 to 1, $\varphi(t, q)$ various continuously from the initial approximation $y_0(t)$, to the exact solution $y(t)$ Such a kind of continous variation is called deformation in homotopy.

By Taylor's theorem, $\varphi(t, q)$ may be expanded in a power series of q as follows:

$$\varphi(t, q) = y_0(t) + \sum_{m=1}^{\infty} y_m q^m \qquad (9)$$

Where

$$y_m(t) = \frac{1}{m!} \frac{\partial^m \varphi(t,q)}{\partial q^m}\Big|_{q=0}. \qquad (10)$$

If the initial guess y_0, auxiliary linear parameter L, the nonzero auxiliary parameter h and the power series (9) of $\varphi(t, q)$ converge at q=1. Then, we have under these assumptions the solution series:

$$y(t) = \varphi(t, 1) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \qquad (11)$$

For brevity, define the vector:

$$\bar{y}_n(t) = \{y_0(t), y_1(t), y_2(t), \dots, y_n(t)\}. \qquad (12)$$

According to equation (11), the governing equation of $y_m(t)$, can be derived from the zero-order deformation equation (6) by differentiating the zero order deformation equation (6) m-times with respective to q and then dividing by m! and finally setting q=0, we have the so called m^{th} –order deformation equation:

$$L[y_m(t) - \chi_m y_{m-1}(t)] = hH(x)R_m(\bar{y}_{m-1}(t))$$
.... (13)

where:

$$R_m(\bar{y}_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1}N[\varphi(t,q)]}{\partial q^{m-1}}\bigg|_{q=0} \ .... (14)$$

And

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation equation (12) is governed by the linear operator L and the term $R_m(\bar{y}_{m-1}(t))$ can be expreesed simply by (13) for any nonlinear operator N.

According to equation (13), the right-hand side of equation (13) is only dependent upon $\bar{y}_{m-1}(t)$. Thus, we gain $y_1(t)$, $y_2(t)$, ... , by means of solving the linear high-order deformation equation (13) one after the other respectively.

The Reliable Algorithm

The homotopy analysis method which provides an analytical approximate solution is applied to various nonlinear problems.

In this section, we shall present a reliable approach of the homotopy analysis method that given in [15].

This modification can be implemented for integer order and fractional order nonlinear equations. To illustrate the basic ideas of this algorithm, we consider the following nonlinear integro-differential equation of fractional order:

$$D_*^\alpha y(t) = N(y) + g(t), \quad t > 0................... (15)$$

Where $m - 1 < \alpha \leq m$, N is a nonlinear operator which might include integer order or fractional order integration, g is a known analytic function and D_*^α is the Caputo fractional derivative of order α.

In view of the homotopy technique, the following homotopy may be constructed:

$$(1 - q)L[\varphi(t,q) - \varphi_0(t)] = qhH(D_*^\alpha \varphi(t,q) - N[\varphi(t,q)] - g(t)). .... (16)$$

Where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auixiliary

function, $\varphi_0(t)$ is an initial guess of $y(t)$ and $L = D_*^\alpha$.

When q=0, equation (16) becomes:
$$L[\varphi(t,0) - \varphi_0(t)] = 0................................ (17)$$

It is obvious that when q=1, equation (16) becomes the original nonlinear equation (15). Thus as q various from 0 to 1, the solution $y(t,q)$ varies from the initial guess y_0 to the solution $\varphi(t,1) = y(t)$. The basic assumption of this approach is that the solution of equation (16) can be expressed as a power series in q, as follows:

$$\varphi = \varphi_0 + q\varphi_1 + q^2\varphi_2 + \ .... (18)$$

Substituting the series (18) into the homotopy (16) and then equating the coefficient of the like powers of q, we obtain the high-order deformation equations,

$$L[\varphi_1] = hH(D_*^\alpha \varphi_0 \text{-} N_0(\varphi_0) - g(t)$$
$$L[\varphi_2] = L[\varphi_1] + hH(D_*^\alpha \varphi_1 \text{-} N_1(\varphi_0, \varphi_1))$$
$$L[\varphi_3] = L[\varphi_2] + hH(D_*^\alpha \varphi_2 \text{-} N_2(\varphi_0, \varphi_1, \varphi_2)).$$
.... (19)
$$L[\varphi_4] = L[\varphi_3] + hH(D_*^\alpha \varphi_3 \text{-} N_3(\varphi_0, \varphi_1, \varphi_2, \varphi_3)) \ \vdots$$

Where
$$N(\varphi_0 + q\varphi_1 + q^2\varphi_2 + \cdots) = N_0(\varphi_0) + qN_1(\varphi_0, \varphi_1) + q^2N_2(\varphi_0, \varphi_1, \varphi_2) \cdots .$$

The approximate solution of equation (15), therefore, can be readily obtained,

$$y(t) = \lim_{q \to 1} \varphi = \varphi_0 + \varphi_1 + \varphi_2 + ... (20)$$

The success of the technique is based on the proper selection of the initial guess φ_0,

Applying the operator I^α to both sides of equation (15) gives,

$$y(t) = \sum_{k=0}^{m-1} y^k(0^+)\frac{t^k}{k!} + I^\alpha N(y) + I^\alpha g(t), t > 0$$
.... (21)

Neglecting the nonlinear term $I^\alpha N(y)$ on the right hand side of equation (21), we can use the remaining part as the initial guess of the solution. That is

$$\varphi_0(t) = \sum_{k=0}^{m-1} y^k(0^+)\frac{t^k}{k!} + I^\alpha g(t) \ldots\ldots\ldots (22)$$

The Reliable Algorithm for Solving Integro-Differential Equations of Fractional Order Using HAM

In this section we shall use the reliable algorithm of the HAM that is given in the last section in order to find the approximate solution of the integro-differential equation of fractional order given by

$$D_*^\alpha y(t) = f(t) + N(y) \ldots\ldots\ldots\ldots\ldots (23)$$

Where $m - 1 < \alpha \le m$, N is a non-linear operator given by

$$N(y) = I^\beta k(y(t)) \ldots\ldots\ldots\ldots (24)$$

Where

$$I^\beta k(y(t)) = \frac{1}{\Gamma(\beta)} \int_0^t (x-s)^{\beta-1} k(y(s)) ds \ldots (25)$$

To find the approximate solution of equation (23) we construct the following homotopy

$$(1-q)L[\varphi(t,q) - \varphi_0(t)] = qhH(D_*^\alpha\varphi(t,q) - N(\varphi(t,q)) - f(t)). \ldots\ldots\ldots\ldots (26)$$

Where $q\epsilon[0,1]$, $h \ne 0$ is a nonzero auxiliary parameter, $H(t) \ne 0$ is an auixilary function, $\varphi_0(t)$ is an initial guess of $y(t)$ and L is an auxiliary linear operator defined as $L = D_*^\alpha$.

Substituting equation (18) into equation (26) and then equating the coefficient of like powers of q, we obtain a high-order deformation equations as given in equations (19) with

$$N_0(\varphi_0) = \frac{1}{\Gamma(\beta)} \int_0^t (x-s)^{\beta-1} A_0(s) ds$$

$$N_1(\varphi_0, \varphi_1) = \frac{1}{\Gamma(\beta)} \int_0^t (x-s)^{\beta-1} A_1(s) ds$$

$$N_2(\varphi_0, \varphi_1, \varphi_2) = \frac{1}{\Gamma(\beta)} \int_0^t (x-s)^{\beta-1} A_2(s) ds$$

$$N_n(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_n)$$
$$= \frac{1}{\Gamma(\beta)} \int_0^t (x-s)^{\beta-1} A_n(s) ds$$

Where
$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} k\left(\sum_{i=0}^n \lambda^i y_i\right)\right]_{\lambda=0}, n = 0,1,2,\ldots$$

And according to equation (22), $\varphi_0(t)$ will be of the form

$$\varphi_0(t) = \sum_{k=0}^{m-1} y^k(0^+)\frac{t^k}{k!} + I^\alpha f(t), \ldots\ldots\ldots (27)$$

Illustrative examples:

In this section, we shall give some problems for linear and nonlinear operators in order to illustrate the validity and accuracy of the proposed method.

Example (1):

Consider the nonlinear fractional order integro-differential equation

$$D_*^{0.75} y(t) = \frac{1}{\Gamma(1.25)} t^{0.25} - \frac{2}{\Gamma(4.5)} t^{3.5} + I^{1.5}(y(t))^2 \ldots\ldots\ldots\ldots\ldots (28).$$

Where $y(0) = 0$, $t \in [0,1]$

The exact solution was given in [9] as $y(t) = t$.

Similar to equation (27) $\varphi_0(t)$ will be of the form

$$\varphi_0(t) = t - \frac{2}{\Gamma(5.25)} t^{4.25}$$

According to the proposed algorithm given in section five with $L = D_*^{0.75}$, we have:

$$L[\varphi_1] = hH(D_*^{0.75}\varphi_0(t) - N_0(\varphi_0)$$
$$- \frac{1}{\Gamma(1.25)} t^{0.25} - \frac{2}{\Gamma(4.5)} t^{3.5})$$

$$L[\varphi_2] = L[\varphi_1] + hH(D_*^{0.75}\varphi_1(t) - N_1(\varphi_0, \varphi_1))$$

$$L[\varphi_3] = L[\varphi_2] + hH(D_*^{0.75}\varphi_2(t) - N_2(\varphi_0, \varphi_1, \varphi_2)) \vdots$$

$$L[\varphi_n] = L[\varphi_{n-1}] + hH(D_*^{0.75}\varphi_{n-1}(t) - N_{n-1}(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_{n-1}))$$

Setting $h = -1$ and $H(t) = 1$, therefore the above equations becomes:

$$\varphi_1(t) = I^{0.75}(N_0(\varphi_0))$$
$$\varphi_2(t) = I^{0.75}(N_1(\varphi_0, \varphi_1))$$
$$\varphi_3(t) = I^{0.75}(N_2(\varphi_0, \varphi_1, \varphi_2))$$
$$\vdots$$
$$\varphi_n(t) = I^{0.75}(N_{n-1}(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_{n-1}))$$

Where
$$N_0(\varphi_0) = \frac{1}{\Gamma(1.5)} \int_0^t (t-s)^{0.5}(\varphi_0(s))^2 ds$$

$$N_1(\varphi_0, \varphi_1) = \frac{1}{\Gamma(1.5)} \int_0^t (t - s)^{0.5} (2\varphi_0(s)\varphi_1(s))ds$$

$$N_2(\varphi_0, \varphi_1, \varphi_2) = \frac{1}{\Gamma(1.5)} \int_0^t (t - s)^{0.5} (2\varphi_0(s)\varphi_2(s) + (\varphi_1(s))^2)ds$$

And so on.

Table (1) represent the comparison between the approximate solution of example (1) using the reliable algorithm of HAM up to three terms and the exact solution.

Table (1)
Comparison of the Approximate Solution of Example (1) using the Reliable HAM with the Exact solution.

t	Approximate Solution using Reliable HAM	Exact Solution
0.1	0.1	0.1
0.2	0.2	0.2
0.3	0.3	0.3
0.4	0.4	0.4
0.5	0.5	0.5
0.6	0.6	0.6
0.7	0.7	0.7
0.8	0.8	0.8
0.9	0.9	0.9

Example (2):

Consider the linear fractional order integro-differential equation

$$D_*^{0.25}y(t) = \frac{6}{\Gamma(3.75)} t^{2.75} - \frac{6}{\Gamma(5.5)} t^{4.5} + I^{1.5}y(t)$$
$$y(0) = 0, \ t \in [0,1]$$

The exact solution was given in [9] as $y(t) = t^3$

Similar to equation (27) $\varphi_0(t)$ will be

$$\varphi_0(t) = t^3 - \frac{6}{\Gamma(5.75)} t^{4.75}$$

And $\varphi_1(t)$, $\varphi_2(t)$,…, will be evaluated according to the following equations

$$L[\varphi_1] = hH(D_*^{0.25}\varphi_0(t) - I^{1.5}(\varphi_0(t)) - \left(\frac{6}{\Gamma(3.75)} t^{2.75} - \frac{6}{\Gamma(5.5)} t^{4.5}\right))$$

$$L[\varphi_2] = L[\varphi_1] + hH(D_*^{0.25}\varphi_1(t) - I^{1.5}(\varphi_1(t)))$$
$$L[\varphi_3] = L[\varphi_2] + hH(D_*^{0.25}\varphi_2(t) - I^{1.5}(\varphi_2(t)))$$
$$\vdots$$
$$L[\varphi_n] = L[\varphi_{n-1}] + hH(D_*^{0.25}\varphi_{n-1}(t) - I^{1.5}(\varphi_{n-1}(t)))$$

Seek $L = D_*^{0.25.}$, $h = -1$ and $H(x) = 1$ then the above equations becomes

$$\varphi_1(t) = I^{1.75}\varphi_0(t)$$
$$\varphi_2(t) = I^{1.75}\varphi_1(t)$$
$$\varphi_3(t) = I^{1.75}\varphi_2(t)$$
$$\vdots$$
$$\varphi_n(t) = I^{1.75}\varphi_{n-1}(t)$$

Following table (2) represent a comparison between the approximate solution of example (2) using the reliable algorithm of HAM up to three terms and the exact solution.

Table (2)
Comparison of the Approximate Solution of Example (2) using the Reliable HAM with the Exact solution.

t	Approximate Solution using Reliable HAM	Exact Solution
0.1	0.001	0.001
0.2	0.008	0.008
0.3	0.027	0.027
0.4	0.064	0.064
0.5	0.125	0.125
0.6	0.216	0.216
0.7	0.343	0.343
0.8	0.512	0.512
0.9	0.729	0.729

Example(3):

Consider the nonlinear fractional integro-differential equation

$$D_*^{0.5}y(t) = \frac{2}{\Gamma(2.5)} t^{1.5} - \frac{5}{\Gamma(5.5)} t^{4.5} + I^{0.5}(y(t))^2$$

Where $y(0) = 0, t \in [0,1]$

The exact solution was given in [9] as $y(t) = t^2$

$\varphi_0(t)$ will be of the form

$$\varphi_0(t) = t^2 - \frac{\Gamma(5)}{\Gamma(6)} t^5$$

As given in equations (19) with $L = D_*^{0.5}$.

$$L[\varphi_1] = hH(D_*^{0.5}\varphi_0(t) - I^{1.5}\varphi_0(t) - N_0(\varphi_0) - (\frac{2}{\Gamma(2.5)}t^{1.5} - \frac{5}{\Gamma(5.5)}t^{4.5}))$$

$$L[\varphi_2] = L[\varphi_1] + hH(D_*^{0.5}\varphi_1(t) - N_1(\varphi_0, \varphi_1))$$

$$L[\varphi_3] = L[\varphi_2] + hH(D_*^{0.5}\varphi_2(t) - N_2(\varphi_0, \varphi_1, \varphi_2)) \vdots$$

$$L[\varphi_n] = L[\varphi_{n-1}] + hH(D_*^{0.5}\varphi_{n-1}(t) - N_{n-1}(\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_{n-1}))$$

Seek $h = -1$ and $H = 1$, therefore the above equations becomes

$$\varphi_1(t) = I^{0.5}(N_0(\varphi_0))$$
$$\varphi_2(t) = I^{0.5}(N_1(\varphi_0, \varphi_1))$$
$$\varphi_3(t) = I^{0.5}(N_2(\varphi_0, \varphi_1, \varphi_2)) \vdots$$
$$\varphi_n(t) = I^{0.5}(N_{n-1}(\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_{n-1}))$$

Where

$$N_0(\varphi_0) = \frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}(\varphi_0(s))^2 ds$$

$$N_1(\varphi_0, \varphi_1) = \frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}(2\varphi_0(s)\varphi_1(s)) ds$$

$$N_2(\varphi_0, \varphi_1, \varphi_2) = \frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}(2\varphi_0(s)\varphi_2(s) + (\varphi_1(s))^2) ds$$

And so on.

Following table (3) represent a comparison between the approximate solution of example (3) the reliable algorithm of HAM up to three terms and the exact solution.

Table (3)
Comparison of the Approximate Solution of Example (3) using the Reliable HAM with the Exact solution.

t	Approximate Solution using Reliable HAM	Exact Solution
0.1	0.01	0.01
0.2	0.04	0.04
0.3	0.09	0.09
0.4	0.16	0.16
0.5	0.25	0.25
0.6	0.36	0.36
0.7	0.49	0.49
0.8	0.639	0.64
0.9	0.806	0.81

Conclusions

Besides, we simply construct the high-order deformation equation of the HAM, we obtain an approximate solutions in power series and these approximate solutions are in good agreement with the exact solution and some times as given in examples one and two gives the exact solution although we are used a few terms. It is remarkable that all methods that used to solve the integro-differential equations of fractional order such as Adomian decomposition method and the variational iteration method may be considered as special cases from the HAM when $h = -1$ and hence it is a special case from the proposed approach.

References
[1] Hilfer R. Editor. "Applications of Fractional Calculus in Physics. Orlando", Academic press, 1999.
[2] Maindari F. "Fractional Calculus: Some Basic Problems in Continuum and Statistical Mechanics. In: Carppinteri A, Mainardi F, Editors. Fractals and Fractional Caluculas in Continuum Mechanics." New York: Springer–Verlag;. pp.291348, 1997.
[3] Oldham KB, spanier J. "The Fractional Calculus". New York: Academic Press; 1974.
[4] Podlubny I. "Fractional Differential Equations". San Diego: Academic Press; 1994.
[5] Mittle R.C. "Solution of Fractional Integro-Differential Equations By Adomian Decomposition Method", International Journal of Applied mathematics and Mechanics, 4(2), pp.87-94, 2008.
[6] Rawashdeh, E.A., "Numercal Solution of Fractional Integro-Differential Equations by Colloections Method'', Applied mathematics and Computations, 176(1) pp.1-6, 2005.
[7] Mohamed, O.H., "Solution of Fractional Integro-Differential Equations by Homotopy Analysis Method"., Journal of Al-Nahrain university, Vol.13 (3), September, pp.149-155, 2010.

[8] Ali S.M., "Some Approximate Solutions of Fractional Integro-Differential Equations", Msc. thesis, College of Science, Al-Nahrian University, 2011.

[9] Wadea, M.H., "Variational Iteration Method for Solving Fractional Order Integro-Differential Equations", Msc. Thesis, college of education, Al-Mustansirya university, 2012.

[10] Liao, S.J., "The Proposed Homotopy Analysis Techniques for the solution of Nonlinear Problems", Ph.D. dissertation, Shanghai Jiao Tong University, 1992.

[11] Liao, S.J., "Beyond Perturbation: Introduction to the Homotopy Analysis Method", Chapman and Hall/CRC press, Boca Raton, 2003.

[12] Jafari H., Seifi S., "Homotopy Analysis Method for Solving Linear and Nonlinear Scinumer" Simulat, 14, pp.2006-12, 2009.

[13] Liao, S.J., "An Approximate Solution Technique which does not dependupon Small Parameters: a Special Example". Int J Nonlinear Mech 30, 37180, 1995.

[14] Liao, S.J., "Homotopy Analysis Method: a new analytic technique for nonlinear problems .Common Nonlinear scinumer simulate, 2(2) 95-100, 1997.

[15] Odibat Z., Momani S., Xu H., "A reliable algorithm of Homotopy Analysis Method for solving nonlinear fractional differential equations", Applied mathematical modeling 34 593-600, 2010.

الخلاصة

في هذا البحث تم استخدام خوارزمية مطورة لطريقة تحليل الهوموتوبي لتقريب حل المعادلات التكاملية-التفاضلية ذات الرتب الكسرية والمتكونة من مشتقة ذات رتبة كسرية وتكامل من الرتبة الكسرية.

المشتقة الكسرية التي تم استخدامها هي من نوع كابوتو والتكامل الكسري من نوع ريمان ليوفيلي. بالأضافة الى ذلك تم اعطاء بعض الأمثلة لتوضيح دقة وفعالية الأسلوب المقترح.