A Note on Pure Submodule Relative to Submodule

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Abstract

In this paper we used the concept of a pure submodule relative to submodule T in two concepts, pure relative to submodule T Baer injective modules and module with pure relative to submodule T intersection property. Some properties and some characterization of this notions are established.

Keywords : pure submodule, T-pure submodule, T-pure Baer injective module.

Introduction

Let R be associative ring with a non-zero identity and R-module will mean unitary left R-module. Recall that a submodule N of an R-module M is pure submodule if for every finitely generated ideal of R, N∩ IM = IN [1]. Following [2] a submodule N of an R-module M is pure submodule relative to submodule T of M (simply T-pure) if N∩IM=IN+T∩(N∩IM) for every ideal I in R. Every pure submodule is T-pure submodule but the converse is not true for example see [2]. An R-module M is called a pure Baer injective module, if for each pure left ideal A of R, any R-homomorphism f : A → M can be extended to an R-hpmomorphism h: R → M [3].

In this paper we introduce the concept of pure relative to submodule Baer injective modules (simply T-pure Baer injective). In [4] modules with the intersection property of any two pure submodule is pure (simply PIP). This led us to introduce the concept of a module with the poperty that the intersection of any two T-pure submodules is T-pure submodule.

1- Pure Relative ToTubmodule Baer Injective Modules.

Now we introduce the concept of pure relative to submodule Baer injective modules (simply T-pure Baer injective).

Definition 1.1 : [2]

Let M be an R-module and T be a submodule of M. A submodule N of M is said to be T-pure if for each ideal I of R, N∩IM = IN +T∩(N∩IM) .

Let T be an ideal in R, a left ideal A of R is said to be T-pure if for every x ∈A there exists y ∈ A such that xy – y ∈ T ∩ A.

Now we give some properties of T-pure submodules.

Remark 1.2 :

1. Let M be an R- module and let N be T-pure submodule of M. If H is T-pure submodule of N, then H is T-pure submodule of M.
2. Let M be an R-module and let N be T-pure submodule of M. if A is a submodule of M containing N, then N is a T-pure submodule of A.
3. Let M be an R- module and let N be T-pure submodule of M. If H is a submodule of N and H is submodule of T, then $\frac{N}{H}$ is T-pure submoduleof $\frac{M}{H}$.
4. Let M be an R –module. Let N and H be submodule of M, If H is T-pure submodule of M and $\frac{N}{H}$ is $\frac{T}{H}$-pure submodule of $\frac{M}{H}$, then N is T-pure submodule of M.

***<u>Proof</u>*:**

1- Let I be an ideal of R, since N is T- pure in M and H is T-pure in N, then N⋂ IM= IN+ T ⋂ (N ⋂ IM) and H ⋂ IN= IH+T⋂ (H ⋂IN) but H ≤ N, therefore
H ⋂IM ⊆ N⋂ IM= IN+T⋂ (N ⋂ IM) and hence H ⋂ IM⊆ [IN+T⋂ (N ⋂ IM)] ⋂ H thus = H⋂ IN+T⋂ (N⋂ IM ⋂ H)= IH + T ⋂ (H⋂ IN) + T⋂ (H ⋂ IM) ⊆ IH +T ⋂ (H ⋂ IM). Since IH + T ⋂ (H ⋂IM) ⊆ H ⋂ IM, then H ⋂ IM = IH + T ⋂ (H ⋂IM).

2-Let I be an ideal of R, since N is T-pure in M, then N ⋂ IM = IN+T ⋂ (N⋂IM).But

$A \leq M$, therefore, $N \cap IA \subseteq N \cap IM = IN + T \cap (N \cap IM)$, and hence $N \cap IA \subseteq [IN + T \cap (N \cap IM)] \cap IA = IN + T \cap (N \cap IA)$. Since $IN + T \cap (N \cap IA) \subseteq N \cap IA$, then $N \cap IA = IN + T \cap (N \cap IA)$.

3-Let I be an ideal of R, since N is T-pure submodule of M, then $N \cap IM = IN + T \cap (N \cap IM)$. So $\frac{N}{H} \cap I\left(\frac{M}{H}\right) = \frac{N}{H} \cap \frac{IM+H}{H} = \frac{(N \cap IM)+H}{H} = \frac{IN+T \cap (N \cap IM)+H}{H}$

$= \frac{IN+H}{H} + \frac{T \cap (N \cap IM)+H}{H} = I\left(\frac{N}{H}\right) + \frac{[T+H] \cap (N \cap IM)}{H}$

$= I\left(\frac{N}{H}\right) + \frac{T+H}{H} \cap \frac{N \cap IM}{H} = I\left(\frac{N}{H}\right) + \frac{T}{H} \cap \left(\frac{N}{H} \cap \frac{IM}{H}\right) = I\left(\frac{N}{H}\right) + \frac{T}{H} \cap \left(\frac{N}{H} \cap I\left(\frac{M}{H}\right)\right)$

4-Clear.

Definition 1.3 :

Let M be an R-module and T be a submodule of M. M is called T-pur Baer injective module if for each T-pure ideal A of R, any R-homomorphism $f : A \rightarrow M$, there exists R-homomrophism $h : R \rightarrow M$ such that $h(a) - f(a) \in T \cap f(A)$ for each $a \in A$.

Clearly, an R-module is pure Baer injective if and only if M is (0)-pure Baer injective. If M is a T_1-pure Baer injective R-module, then M is T_2-pure Baer injective for each submodule T_2 containing T_1, Thus every pure Baer injective module is T-pure Baer injective R-module.

Now we give anther carectrization of T-pure Baer injective modules.

Theorem 1.4 :

For an R-module M the following are equivalent:

1- M is T-pure Baer injective,
2- For T-pure left ideal A of R and every R-homomorphism $f : A \rightarrow M$ there exists $m \in M$ such that for all $a \in A$, $am - f(a) \in T \cap f(A)$,

<u>Proof</u>: Clear

Proposition 1.5 :

If the direct product $\prod_{\alpha \in \Lambda} M_\alpha$ of R-modules is $J_\alpha(T)$-pure Baer injective, where J_α is pinjection of M_α into $\prod_{\alpha \in \Lambda} M_\alpha$ then M_α is T-pure Baer injective for each α.

<u>Proof</u>:

Let A be a T-pure submodule of M_α and $f : A \rightarrow M_\alpha$ be R-homomorphism. Since $\prod_{\alpha \in \Lambda} M_\alpha$ is $J_\alpha(T)$--pure Baer injective, therefore there is an R-homomorphism $h : R \rightarrow \prod_{\alpha \in \Lambda} M_\alpha$ such that $h \circ i(a) - J_\alpha \circ f(a) \in J_\alpha(T) \cap J_\alpha \circ f(A)$, thus $\rho_\alpha \circ h\, i(a) - \rho_\alpha \circ J_\alpha \circ f(a) \in \rho_\alpha \circ J_\alpha(T) \cap \rho_\alpha \circ J_\alpha \circ f(A)$ where ρ_α is the projection map. Put $h_1 = \rho_\alpha \circ h$, then $h_1 \circ i(a) - f(a) \in T \cap f(A)$. Hence M_α is T-pure Baer injective module for each α.

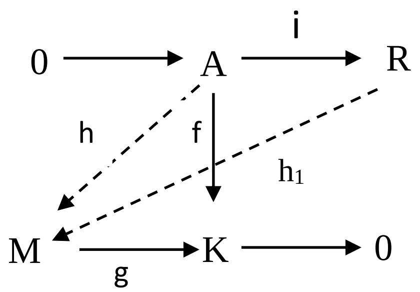
Recall that an R-module P is projective, if given any R-epimorphismf $: A \rightarrow B$, nay R-homomorphism $g : M \rightarrow B$ can be lifted to an R-homomorphism $h : M \rightarrow A$ [5].

Theorem 1.6 :

If every T-pure ideal of R is projective. Then the homomorphic image of a T-pure Baer injective module is t-pure Baer injective.

<u>Proof:</u>

Consider the following diagram of R-modules



Where A is left T-pure ideal of R and i is the inclusion map and M is T-pure Baer injective module. Projectivity of A shows that for some R-homomorphism $h : A \rightarrow M$ ther is R-homomorphism $h: A \rightarrow M$ such that $f = gh$. Since M is T-pure Baer injective module, there exists $h_1: R \rightarrow M$ such that $h_1 \circ i(a) - h(a) \in T \cap h(A)$ for all $a \in A$, thus $g \circ h_1 \circ i(a) - g\, h(a) \in g(T) \cap g \circ h(A)$. Put $h^* = g \circ h_1$, hence $h^* \circ i(a) - f(a) \in T \cap f(A)$. Therefore K is T-pure Baer injective.

The converse of the above theorem is not true in general. We need the following concept, let M be an R-module and T a submodule of M, M is said to be projective relative to submodule T (simply T-projective), if for each R-epimorphism $f : A \rightarrow B$, nay R-homomorphism $g : M \rightarrow B$ there exists

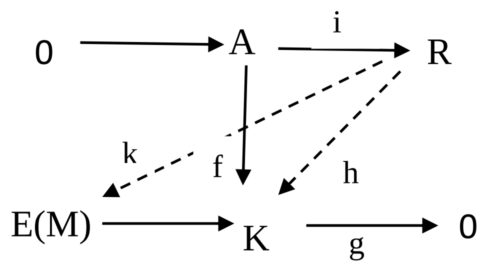
R-homomorphism h : M → A such that f∘h(m) – g(m) ∈g(T) for each m in M [2].

Theorem 1.7:

If the homomorphic image of injective module is T-pure Baer injective. Then every T-pure ideal of R is T-projective.

***Proof*:**

Let A be a T-pure left ideal of R and M be an R-module whose injective hull is E(M). Consider the following diagram:



K is T-pure Baer injective by assumption. So there exists an R-homomorphism h: R → K such that h∘ i(a) – f(a) ∈ T ∩ f(A). Since R is projective, there exists k : R → E(M) such that g ∘ k = h, and so g ∘ $k \circ i(a)$∈ T ∩ f(A) ⊆ f(A). Put h^{*}= $k \circ i$, thus g ∘ h^{*}(a) – f(a) ∈ f(A). Hence A is T-projective.

2-Modules with T-Pure Intersection Property

In this section Let R be commutative ring with identity, we introduce the concept of module which have T-pure intersection property.

Definition 2.1 :

An R-module M is said to have the pure relative to submodule intersection property (for short T- PIP) if the intersection of any two T- pure submodules is again T- pure.

Proposition 2.2:

1. If an R- module M has the T- PIP, then every T- pure submodule of M has the T- PIP.
2. Let N be T- pure submodule of an R- module M and T submodule of N. M has T- PIP if and only if $\frac{M}{N}$ has $\frac{T}{N}$-PIP.

***Proof* :**

1- Clear.

2- (⟹).

Let $\frac{A}{N}$, $\frac{B}{N}$ be two $\frac{T}{N}$- pure submodules of $\frac{M}{N}$ and let K be an ideal in R. We want to show that

$(\frac{A}{N}\cap\frac{B}{N}) \cap K(\frac{M}{N}) = K(\frac{A}{N}\cap\frac{B}{N})+\frac{T}{N}\cap[(\frac{A}{N}\cap\frac{B}{N})\cap K(\frac{M}{N})]$

We claim that each of A and B is T-pure in M. To show this, let I be an ideal in R and let x ∈A⋂IM. Since $\frac{A}{N}$ is $\frac{T}{N}$-pure in $\frac{M}{N}$, then $\frac{A}{N}\cap I(\frac{M}{N}) = I(\frac{A}{N}) +\frac{T}{N}\cap(\frac{A}{N}\cap I(\frac{M}{N}))$,thus $\frac{A}{N}\cap\frac{IM+N}{N} = \frac{IA+N}{N}+\frac{T}{N}\cap(\frac{A}{N}\cap(\frac{IM+N}{N}))$, and this implies that

$\frac{A\cap(IM+N)}{N}= \frac{IA+N}{N}+ \frac{(T+N)\cap(A\cap(IM+N))}{N} = \frac{(IA+N)+T+N)\cap(A\cap(IM+N))}{N}$, therefore,

A ⋂(IM+N) = IA+T ⋂(A⋂IM) + N, and hence (A ⋂IM) + N = IA + T⋂ (A ⋂IM) +N

Since x∈ A⋂IM ⊆ A⋂ (IM+N), then x∈IA + T ⋂(A ⋂IM) + N

Let x = w + m + n, where w ∈ IA and m ∈ T ⋂ (A⋂IM) and n ∈N

Now, consider n = x – w – m ∈ N⋂ IM = IN + T ⋂(N ⋂IM) ⊆ IA + T ⋂ (A ⋂IM)

And hence A is T- pure in M. Since M has the T-PIP, then A ⋂B is T- pure in M.

Thus (A ⋂B) ⋂ KM = K (A ⋂B) + T ⋂ ((A ⋂B) ⋂IM)

Now, let x∈$(\frac{A}{N}\cap\frac{B}{N}) \cap K(\frac{M}{N})$, then x = w +N, where w ∈KM and x = a +N = b +N, where a ∈A and b ∈B. Thus w- a ∈ N ⊆ A, w – b ∈ N ⊆ B and hence w ∈ A⋂B. Thus w ∈ (A⋂ B) ⋂ KM = K (A⋂B) + T ⋂((A ⋂B) ⋂KM). Then x = w + N ∈ K ($\frac{A\cap B}{N}$) = $K(\frac{A}{N}\cap\frac{B}{N}) \subseteq K(\frac{A}{N}\cap\frac{B}{N}) + \frac{T}{N}\cap((\frac{A}{N}\cap\frac{B}{N})\cap K\frac{M}{N})$

(⟸) Conversely let E and F be T- pure submodule of M, let N be a submodule of E and N be a submodule of F then $\frac{E}{N}$ and $\frac{F}{N}$ is $\frac{T}{N}$-pure submodule of $\frac{M}{N}$. Since $\frac{M}{N}$ has $\frac{T}{N}$-PIP, then $\frac{E}{N} \cap \frac{F}{N} = \frac{E \cap F}{N}$ is $\frac{T}{N}$- pure submodule of $\frac{M}{N}$. Therefore $E \cap F$ is T- pure submodule of M.

Theorem 2.3:

Let M be an R- module, then M has the T-PIP if and only if
$(IA \cap IB) + T \cap ((A \cap B) \cap IM) = I(A \cap B) + T \cap ((A \cap B) \cap IM)$ for every ideal I of R and for every T- pure submodule A and B of M.

***Proof*:**

Suppose M has the T-PIP then for each T-pure submodules A and B, $A \cap B$ is T-pure. Let I be an ideal in R, then
$(A \cap B) \cap IM = I(A \cap B) + T \cap ((A \cap B) \cap IM)$.
It is clear that $I(A \cap B) + T((A \cap B) \cap IM) \subseteq (IA \cap IB) + T((A \cap B) \cap IM)$. But $(IA \cap IB) + T \cap ((A \cap B) \cap IM) \subseteq A \cap (B \cap IM) = (A \cap B) \cap IM = I(A \cap B) + T \cap ((A \cap B) \cap IM)$. Thus $IA \cap IB + J(R)M \cap ((A \cap B) \cap IM) = I(A \cap B) + T \cap ((A \cap B) \cap IM)$.
Conversely, let A and B be T-pure submodule of M and I an ideal in R. then $A \cap B \cap IM = A \cap (B \cap IM) = A \cap (IB + T \cap (B \cap IM))$. Similarly $A \cap B \cap IM = B \cap (IA + T \cap (B \cap IM))$. But A, B are T- pure in M. Thus $A \cap B \cap IM \subseteq IA \cap IB + T \cap (A \cap B \cap IM)$
$= I(A \cap B) + T(A \cap B \cap IM)$

Theorem 2.4:

Let M be an R- module, then M has the T-PIP if and only if for every T-pure submodules A and B of M and for every R- homomorphism $f = A \cap B \longrightarrow M$ such that $(A \cap \mathrm{Im}\, f) + T \cap (A + \mathrm{Im}\, f \cap IM) = \{0\}$ and $A + \mathrm{Im}\, f$ is T- pure in M, ker f is T- pure in M.

***Proof*:**

Assume that M has the T-PIP. Let A and B be T-pure submodules of M and $f = A \cap B \longrightarrow M$ be an R-homomorphism such that $A \cap \mathrm{Im} f = \{0\}$ and $A + \mathrm{Im}\, f$ is T- pure in M.
Let $K = \{x + f(x), x \in A \cap B\}$. It is clear that K is a submodule of M.
To show that K is T- pure in M. let I be an ideal in R and

$y = \sum_{i=1}^{n} r_i m_i \in K \cap IM$, $r_i \in R$, $m_i \in M$.
Hence $y = \sum_{i=1}^{n} r_i m_i = x + f(x)$ for some $x \in A \cap B$. Since $y = \sum_{i=1}^{n} r_i m_i = x + f(x) \in A \cap B + \mathrm{Im}\, f \subseteq A + \mathrm{Im}\, f$ and $A + \mathrm{Im}\, f$ is T- pure in M. Thus $y = \sum_{i=1}^{n} r_i m_i \in (A + \mathrm{Im}\, f) \cap IM = I(A + \mathrm{Im}\, f) + T \cap ((A + \mathrm{Im}\, f) \cap IM)$ Therefore $\sum_{i=1}^{n} r_i m_i = \sum_{i=1}^{n} r_i (x_i + y_i) + k$, $x_i \in A$, $y_i \in \mathrm{Im}\, f$, $\forall i = 1, \ldots, n$ $k \in T \cap (A + \mathrm{Im}\, f \cap IM)$. Thus $y = \sum_{i=1}^{n} r_i m_i = \sum_{i=1}^{n} r_i x_i + \sum_{i=1}^{n} r_i y_i + k$, hence $x - \sum_{i=1}^{n} r_i x_i = \sum_{i=1}^{n} r_i y_i - f(x) + k \in (A \cap \mathrm{Im}\, f) + T \cap ((A + \mathrm{Im}\, f) \cap IM) = 0$. Therefore $x = \sum_{i=1}^{n} r_i x_i \in (A \cap B) \cap IA$.
But $A \cap B$ is T- pure in M, hence is T-pure in A. Thus $(A \cap B) \cap IA = I(A \cap B) + T \cap ((A \cap B) \cap IA)$ by theorem (2.3). Thus $x \in I(A \cap B) + T \cap ((A \cap B) \cap IA)$. Let $x = \sum_{i=1}^{n} r_i w_i + h$, $w_i \in A \cap B$, $h \in T \cap ((A \cap B) \cap IA)$. Then $f(x) = \sum_{i=1}^{n} r_i f(w_i) + f(h)$. Now $y = x + f(x) = \sum_{i=1}^{n} r_i w_i + \sum_{i=1}^{n} r_i f(w_i) + f(h) = \sum_{i=1}^{n} r_i (w_i + f(w_i)) + f(h) \in IK + T \cap (K \cap IM)$. Thus $K \cap IM = IK + T \cap (K \cap IM)$ and K is T- pure in M. Next we show that $\ker f = (A \cap B) \cap K$. Let $x \in \ker f$, then $x \in A \cap B$ and $f(x) = 0$. Hence $x \in K$, Now let $x \in (A \cap B) \cap K$, then $x = y + f(y)$, $y \in A \cap B$, then $x - y = f(y) \in A \cap \mathrm{Im}\, f \leq (A \cap \mathrm{Im}\, f) + T \cap (A + \mathrm{Im}\, f \cap IM) = 0$. Therefore

$f(x) = f(y) = 0$ and $x \in \ker f$. Since M has T- PIP, then $(A \cap B) \cap K = \ker f$ is T- pure in M. Conversely, let A and B be T- pure submodules of M. Define $f = A \cap B \rightarrow M$ by $f(x) = 0$, $\forall x \in A \cap B$. It is clear $(A \cap \mathrm{Im}\, f) + T \cap (A + \mathrm{Im}\, f \cap IM) = 0$ and $A + \mathrm{Im}\, f = A$ is T-pure in M, then $\ker f = A \cap B$ is T–pure in M.

By the same argument one can prove the following

Theorem 2.5:

Let M be an R- module, then M has the T-PIP if and only if for every T-pure submodules A and B of M and for every R-homomorphism $f = A \cap B \rightarrow C$, where C is a submodule of M such that $A \cap C + T \cap (A + C \cap IM) = 0$ and A + C is T- pure in M, kerf is T-pure in M.

References

[1] Faith C., Algebra I, rings, "Modules and Categories", Springer Verlag, Berline, Heidelberg, Newyork 1973.

[2] Mehdi S. Abbas, "Purity and Projectivity Relative to A Submodule" To appear.

[3] Nada M. Al Thani, "Pure Baer Injective Modules" Internat. J. Math. &Math. Sci. vol. 20, No. 3, 529-538, 1997.

[4] B.H Al- Bahraany, "Modules with the Pure Intersection Property". PhD. Thesis, College of Science, University of Baghdad, 2000.

[5] Kasch F., Modules and Rings, Academic

الخلاصة

في هذا العمل استخدم مفهوم المقاس الجزئي النقي بالنسبة الى مقاس جزئي في استحداث مفهومين احدهم المقاسات بير اغماري النقي بالنسبة الى مقاس جزئي و المقاسات التي تمتلك خاصية التقاطع النقي بالنسبة الى مقاس جزئي. تم دراست بعض الخواص التصنيفات بالنسبة الى هاتين المفهومين.