

Solving Fuzzy Fractional Boundary Value Problems Using Fractional Differential Transform Method

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Abstract

In this paper we shall present an approximate solution for fuzzy fractional boundary value problems (FFBVP's) based on the Fractional Differential Transform Method (FDTM) which is proposed to solve linear and nonlinear FFBVP's. The fuzziness will appear in the boundary conditions, to be fuzzy numbers. The solution of our model equations are calculated in the form of convergent series with easily computable components. Some examples are solved as an illustrations, the numerical results shows that the followed approach is easy to implement and accurate when applied to FFBVP's.

Keywords: Fractional Calculus, Fuzzy Differential Equations, Fractional Differential Transform Method.

Introduction

The subject of fractional calculus (that is calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables [1].

Numerous publications have been focused on analytical and numerical study of the fractional initial value problems (FIVP_s) (see [2], [3], [4], [5], [6], [7], [8], [9]).

Comparatively, little attention has been paid to the fractional two point boundary value problems (FBVP_s). In the context the approximate solution of the fuzzy fractional boundary value problems will be investigated. The concept of fuzzy sets which was originally introduced by Zadeh [10] led to the definition of the fuzzy number and its implementation in fuzzy control and approximate reasoning problems. The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka [11], Nahmias and Ralescu [12,13] all of which observed the fuzzy number as a collection of α -levels, $0 < \alpha \leq 1$. The basic concepts on fuzzy sets, fuzzy

differentials and fuzzy differential equations can be found in [14,15,16,17,18]. The study of fuzzy differential equations (FDEs) forms a suitable setting for the mathematical modeling of real world problems in which uncertainty or vagueness pervades.

The proposed method that will be used to approximate the fuzzy fractional boundary value problems is the differential transform method (DTM). The concept of differential transform was first proposed by Zhou [19] and it was applied to solve linear and non-linear initial value problems in electric circuit analysis. This method provides an iterative procedure to obtain the spectrum of the analytical solutions.

The DTM has been successfully applied to a wide class of differential equations arising in many areas of science and engineering. Since many physical phenomena are more faithfully modeled by fractional differential equations (FDEs), Arikoglu and Ozkol [20], developed the fractional differential transform method (FDTM) for their efficient solution. Also, Odibat and Shawagfeh [21] suggested the same technique as a generalized Taylor's formula for solving FDEs. The differential transformation method evaluates the approximate solution by the finite Taylor series. But, in the differential transformation method, the derivative is not computed directly; instead, the relative derivatives are calculated by an iteration procedure. New equations are obtained from the original

equations by applying differential transformations. This paper is organized as follows:

In section two, we recall the basic concept of fuzzy set theory, in section three, definitions of Riemann-Liouville and Caputo fractional derivatives are considered, fractional differential transform method and some of its related theorems are given in section four, our approach is presented in section five, and two numerical examples are solved in section six to illustrate the ability of the FDTM.

Basic Concepts of Fuzzy Set Theory

In this section, we shall present some basic definitions of fuzzy set theory including the definition of fuzzy numbers and fuzzy functions.

Definition (1), [22]

Let X be any set of elements, a fuzzy set \tilde{A} is characterized by a membership function $\mu_{\tilde{A}}(x): X \rightarrow [0, 1]$, and may be written as the set of points

$$\tilde{A} = \{ (x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1 \}.$$

Definition (2), [22]:

The crisp set of elements that belong to the set \tilde{A} at least to the degree α is called the weak α -level set (or weak α -cut), and is defined by:

$$A_{\alpha} = \{ x \in X : \mu_{\tilde{A}}(x) \geq \alpha \}$$

While the strong α -level set (or strong α -cut) is defined by:

$$A'_{\alpha} = \{ x \in X : \mu_{\tilde{A}}(x) > \alpha \}$$

Definition (3), [10]:

A fuzzy subset \tilde{A} of a universal space X is convex if and only if the sets A_{α} are convex, $\forall \alpha \in [0, 1]$.

or equivalently, we can define convex fuzzy set directly by using its membership function to satisfy:

$$\mu_{\tilde{A}}[\lambda x_1 + (1-\lambda)x_2] \geq \text{Min} \{ \mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2) \}$$

for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$.

Remark (1), [23]:

A fuzzy number \tilde{M} may be uniquely represented in terms of its α -level sets, as the following closed intervals of the real line:

$$M_{\alpha} = [m - \sqrt{1-\alpha}, m + \sqrt{1-\alpha}]$$

or

$$M_{\alpha} = [\alpha m, \frac{1}{\alpha} m]$$

Where m is the mean value of \tilde{M} and $\alpha \in (0, 1]$. This fuzzy number may be written as $\tilde{M} = [\underline{M}, \overline{M}]$, where \underline{M} refers to the greatest lower bound of \tilde{M} and \overline{M} to the least upper bound of \tilde{M} .

Remark (2), [23]:

Similar to the second approach given in remark (1), one can fuzzyfy any crisp or nonfuzzy function f, by letting:

$$\underline{f}(x) = \alpha f(x), \overline{f}(x) = \frac{1}{\alpha} f(x), x \in X, \alpha \in (0, 1],$$

and hence the fuzzy function \tilde{f} in terms of its β -levels is given by $f_{\alpha} = [\underline{f}, \overline{f}]$.

Riemann-Liouville and Caputo Fractional Derivatives [24]

There are various types of definitions, for the fractional derivatives of order $q > 0$, the most commonly used definitions among various definitions of fractional derivatives of order $q > 0$ are the Riemann-Liouville and Caputo formula. Ones which use fractional integrals and derivatives of the whole order. The difference between the two definitions in the order of evaluation. Riemann-Liouville fractional integration of order q is defined as:

$$J_{x_0}^q f(x) = \frac{1}{\Gamma(q)} \int_{x_0}^x (x-t)^{q-1} f(t) dt, q > 0, x > 0 \dots (1)$$

The following equations defined Riemann-Liouville and Caputo fractional derivatives of order q, respectively:

$$D_{x_0}^q f(x) = \frac{d^m}{dx^m} \left[J_{x_0}^{m-q} f(x) \right] \dots \dots \dots (2)$$

$$D_{*x_0}^q f(x) = J_{x_0}^{m-q} \left[\frac{d^m}{dx^m} f(x) \right]$$

Where $m-1 \leq q < m$ and $m \in \mathbb{Z}^+$, $x > x_0$. From (1) and (2), we have:

$$D_{x_0}^q f(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{x_0}^x (x-t)^{m-q-1} f(t) dt, x > x_0. \dots \dots \dots (3)$$

Fractional Differential Transform Method [25]

In this section, we shall introduce the fractional differential transform method used in this paper to obtain an approximate solutions for the fuzzy fractional boundary value problems. This method has been developed in [20] as follows:

Let us expand the analytical and continuous function $f(x)$ in terms of a fractional power series as follows:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^{\frac{k}{\beta}}, \dots\dots\dots (4)$$

Where β is the order of fraction and $F(k)$ is the fractional differential transform of $f(x)$. In order to avoid fractional initial and boundary conditions, we shall define the fractional derivative in the Caputo sense. The relation between the Riemann–Liouville operator and Caputo operator is given by:

$$D_{x_0}^q f(x) = D_{x_0}^q \left[f(x) - \sum_{k=0}^{m-1} \frac{1}{k!} (x-x_0)^k f^{(k)}(x_0) \right] \dots\dots\dots (5)$$

Setting,

$$f(x) = f(x) - \sum_{k=0}^{m-1} \frac{1}{k!} (x-x_0)^k f^{(k)}(x_0)$$

In Eq.(3) and using Eq.(5), we obtain fractional derivative in the Caputo sense [26] as follows:

$$D_{x_0}^q f(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \left\{ \int_{x_0}^x \frac{f(x) - \sum_{k=0}^{m-1} \left(\frac{1}{k!}\right) (x-x_0)^k f^{(k)}(x_0)}{(x-t)^{1+q-m}} dt \right\} \dots\dots\dots (6)$$

Since the initial conditions are implemented to the integer order derivatives, the transformation of the initial conditions are defined as follows:

$$F(k) = \begin{cases} \frac{1}{\left(\frac{k}{\beta}\right)!} \left(\frac{d^{\frac{k}{\beta}} f(x)}{dx^{\frac{k}{\beta}}} \right)_{x=x_0} & , \text{if } \frac{k}{\beta} \in \mathbb{Z}^+ \\ 0 & , \text{if } \frac{k}{\beta} \notin \mathbb{Z}^+ \end{cases} , k=0,1,2,\dots,(\beta q-1).$$

where q is the order of the fractional differential equation considered.

Theorem (1), [27]

If $f(x)=g(x) \pm h(x)$, then $F(k)= G(k) \pm H(k)$, where F, G and H are the differential transforms of f, g and h , respectively.

Theorem (2), [27]:

If $f(x) =g(x)h(x)$, then

$$F(k) = \sum_{\ell=0}^k G(\ell) H(k-\ell), \text{ where } F, G \text{ and } H \text{ are}$$

the differential transforms of f, g and h , respectively.

Theorem (3), [27]

Suppose that F, G_1, G_2, \dots, G_n are the differential transforms of f, g_1, g_2, \dots, g_n , respectively; then we have:

If $f(x)=g_1(x) g_2(x) \dots g_{n-1}(x) g_n(x)$, then

$$F(k) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}-1} \dots \sum_{k_2=0}^{k_3-1} \sum_{k_1=0}^{k_2-1} G_1(k_1)$$

$$G(k_2-k_1) \dots G_{n-1}(k_{n-1}-k_{n-2}) G_n(k-k_{n-1})$$

Theorem (4), [27]

$f(x)=(x-x_0)^q$, then $F(k)=\delta(k-\beta q)$,

where:

$$\delta(k) = \begin{cases} 1, & \text{if } k=0 \\ 0, & \text{if } k \neq 0 \end{cases}$$

Theorem (5), [27]

If $f(x)=D_{x_0}^q [g(x)]$, then

$$F(k) = \frac{\Gamma(q+1+k/\beta)}{\Gamma(1+k/\beta)} G(k+\beta q).$$

Theorem (6), [27]

For the production of fractional derivatives in the most general form:

$$f(x) = \frac{d^{\mathcal{A}}}{dx^{\mathcal{A}}} [g_1(x)] \frac{d^{\mathcal{B}}}{dx^{\mathcal{B}}} [g_2(x)] \dots \frac{d^{\mathcal{A}_{n-1}}}{dx^{\mathcal{A}_{n-1}}} [g_{n-1}(x)] \frac{d^{\mathcal{A}_n}}{dx^{\mathcal{A}_n}} [g_n(x)],$$

then,

$$R(k) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \dots \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}}$$

$$\frac{\Gamma(q_1+1+k_1/\beta)}{\Gamma(1+k_1/\beta)} \frac{\Gamma(q_1+1+(k_2-k_1)/\beta)}{\Gamma(1+(k_2-k_1)/\beta)} \dots$$

$$\dots \frac{\Gamma(q_{n-1}+1+(k_{n-1}-k_{n-2})/\beta)}{\Gamma(1+(k_{n-1}-k_{n-2})/\beta)}$$

$$\frac{\Gamma(q_n+1+(k-k_{n-1})/\beta)}{\Gamma(1+(k-k_{n-1})/\beta)} G_1(k_1+\beta q_1) \times G_2(k_2-k_1+\beta q_2)$$

$$\dots G_{n-1}(k_{n-1}-k_{n-2}+\beta q_{n-1}) G_n(k-k_{n-1}+\beta q_n),$$

where $\beta q_i \in Z^+$ for $i=1,2,3,\dots,n$.

The Proposed Approach

In this paper we shall consider the following fuzzy boundary value problem:

$$D_*^\gamma y(x) + g(t, y(x), D_*^\theta y(x)) = 0, \quad a < x < b, \quad 1 < \gamma \leq 2$$

$$, 0 < \theta \leq 1$$

$$y(a) = a, \quad y(b) = b.$$

Where a and b are considered to be fuzzy numbers and therefore the solution of course (which is ensure from the existence and uniqueness theorem of fuzzy ordinary differential equation, see [28]), will be fuzzy function having the form $y = [\underline{y}, \bar{y}]$ where \bar{y} represent the lower solution and \underline{y} represent the upper solution, and in order to find \underline{y} and \bar{y} we must solve the following problems respectively:

$$D_*^\gamma \underline{y}(x) + g(t, \underline{y}(x), D_*^\theta \underline{y}(x)) = 0.$$

where

$$\underline{y}(a) = a - \sqrt{1-\alpha}, \quad \underline{y}(b) = b - \sqrt{1-\alpha}$$

and

$$D_*^\gamma \bar{y}(x) + g(t, \bar{y}(x), D_*^\theta \bar{y}(x)) = 0.$$

where

$$\bar{y}(a) = a + \sqrt{1-\alpha}, \quad \bar{y}(b) = b + \sqrt{1-\alpha}$$

By using the fractional differential transforms method, and hence we get our desired solution.

Numerical Examples

In this section, two illustrative examples are given in order to illustrate the proposed method.

Example (1):

Consider the following linear FFBVP:

$$D_x^{\frac{3}{2}} y(x) + \frac{1}{3} y(x) + \frac{1}{4} D_x^{\frac{1}{2}} y(x) = R(x), \quad 0 < x < 1$$

..... (7)

With fuzzy boundary conditions:

$$y(0) = 0, \quad y(1) = 0$$

Where

$$R(x) = \frac{-7}{2\sqrt{\pi}} x^2 + \frac{x}{3} - \frac{2}{3\sqrt{\pi}} x^{\frac{3}{2}} - \frac{x^2}{3}.$$

The solution will be of the form $[y \bar{y}]$. Now to find \underline{y} , we must solve the problem:

$$D_x^{\frac{3}{2}} \underline{y}(x) + \frac{1}{3} \underline{y}(x) + \frac{1}{4} D_x^{\frac{1}{2}} \underline{y}(x) = R(x) \quad \dots \dots \dots (8)$$

With the boundary conditions:

$$\underline{y}(0) = -\sqrt{1-\alpha} \quad \text{and} \quad \underline{y}(1) = -\sqrt{1-\alpha} \quad \dots \dots \dots (9)$$

Applying the differential transform method for the problems (8) and (9) and according to theorem (5) we have:

$$\frac{\Gamma\left(\frac{5+k}{2}\right)}{\Gamma\left(\frac{2+k}{2}\right)} \underline{Y}(k+3) + \frac{1}{3} \underline{Y}(k) + \frac{\Gamma\left(\frac{3+k}{2}\right)}{4\Gamma\left(\frac{2+k}{2}\right)} \underline{Y}(k+1) = R(k)$$

or equivalently,

$$\underline{Y}(k+3) = \frac{\Gamma\left(\frac{2+k}{2}\right)}{\Gamma\left(\frac{5+k}{2}\right)} R(k) - \frac{1}{3} \frac{\Gamma\left(\frac{2+k}{2}\right)}{\Gamma\left(\frac{5+k}{2}\right)} \underline{Y}(k) -$$

$$- \frac{\Gamma\left(\frac{3+k}{2}\right)}{4\Gamma\left(\frac{2+k}{2}\right)} \underline{Y}(k+1) \quad \dots \dots \dots (10)$$

Where

$$R(k) = \frac{-7}{2\sqrt{\pi}} \delta(k-1) + \frac{1}{3} \delta(k-2) - \frac{2}{3\sqrt{\pi}} \delta(k-3)$$

$$- \frac{1}{3} \delta(k-4)$$

..... (11)

With the boundary conditions:

$\underline{Y}(0)=-\sqrt{1-\alpha}$; $\underline{Y}(1)=0$, $\underline{Y}(2)=y'(0)=a$,
Hence,

$$\begin{aligned} \underline{Y}(3) &= \frac{\sqrt{1-\alpha}}{3\Gamma(5/2)}, \quad \underline{Y}(4) = \frac{-7\Gamma(3/2)}{2\Gamma(3)\sqrt{\pi}} - \frac{\Gamma(2)}{4\Gamma(3)} a, \\ \underline{Y}(5) &= \frac{\Gamma(2)}{3\Gamma(7/2)} - \frac{\Gamma(2)}{3\Gamma(7/2)} a - \frac{\sqrt{1-\alpha}}{12\Gamma(7/2)}, \\ \underline{Y}(6) &= \frac{-2\Gamma(5/2)}{3\sqrt{\pi}\Gamma(4)} - \frac{\sqrt{1-\alpha}}{9\Gamma(4)} + \frac{7\Gamma(3/2)}{8\sqrt{\pi}\Gamma(4)} + \frac{\Gamma(2)}{16\Gamma(4)} a, \\ \underline{Y}(7) &= \frac{-\Gamma(3)}{3\Gamma(9/2)} + \frac{7\Gamma(3/2)}{6\Gamma(9/2)\sqrt{\pi}} + \frac{\Gamma(2)}{6\Gamma(9/2)} a - \\ &\quad - \frac{\Gamma(2)}{12\Gamma(9/2)} + \frac{\sqrt{1-\alpha}}{48\Gamma(9/2)} \end{aligned}$$

Then the approximate solution for \underline{y} up to eight-terms is:

$$\underline{y}(x) = \sum_{k=0}^7 \underline{Y}(k) x^{\frac{k}{2}} = \underline{Y}(0)x^0 + \underline{Y}(1)x^{1/2} + \underline{Y}(2)x + \underline{Y}(3)x^{3/2} + \underline{Y}(4)x^2 + \underline{Y}(5)x^{5/2} + \underline{Y}(6)x^3 + \underline{Y}(7)x^{7/2}$$

Therefore;

$$\begin{aligned} \underline{y}(x) &= -\sqrt{1-\alpha} + ax + \frac{\sqrt{1-\alpha}}{3\Gamma(5/2)} x^{3/2} - \left(\frac{7\Gamma(3/2)}{2\Gamma(3)\sqrt{\pi}} + \frac{\Gamma(2)}{4\Gamma(3)} a \right) x^2 \\ &\quad + \left(\frac{\Gamma(2)}{3\Gamma(7/2)} - \frac{\Gamma(2)}{3\Gamma(7/2)} a - \frac{\sqrt{1-\alpha}}{12\Gamma(7/2)} \right) x^{5/2} + \left(\frac{-2\Gamma(5/2)}{3\sqrt{\pi}\Gamma(4)} - \right. \\ &\quad \left. - \frac{\sqrt{1-\alpha}}{9\Gamma(4)} + \frac{7\Gamma(3/2)}{8\sqrt{\pi}\Gamma(4)} + \frac{\Gamma(2)}{16\Gamma(4)} a \right) x^3 + \left(\frac{-\Gamma(3)}{3\Gamma(9/2)} + \frac{7\Gamma(3/2)}{6\Gamma(9/2)\sqrt{\pi}} \right. \\ &\quad \left. + \frac{\Gamma(2)}{6\Gamma(9/2)} a - \frac{\Gamma(2)}{12\Gamma(9/2)} + \frac{\sqrt{1-\alpha}}{48\Gamma(9/2)} \right) x^{7/2}. \end{aligned}$$

and in order to find the constant a one can use $\underline{y}(1)=-\sqrt{1-\alpha}$ then we shall find a according to the values of α .

In similar manner one can apply the same approach given for the lower case to find the upper solution as follows:

$$\begin{aligned} \frac{\Gamma\left(\frac{5+k}{2}\right)}{\Gamma\left(\frac{2+k}{2}\right)} Y(k+3) + \frac{1}{3} Y(k) + \frac{\Gamma\left(\frac{3+k}{2}\right)}{4\Gamma\left(\frac{2+k}{2}\right)} Y(k+1) &= R(k) \\ Y(k+3) &= \frac{\Gamma\left(\frac{2+k}{2}\right)}{\Gamma\left(\frac{5+k}{2}\right)} R(k) - \frac{1}{3} \frac{\Gamma\left(\frac{2+k}{2}\right)}{\Gamma\left(\frac{5+k}{2}\right)} Y(k) - \\ &\quad - \frac{\Gamma\left(\frac{3+k}{2}\right)}{4\Gamma\left(\frac{2+k}{2}\right)} Y(k+1) \end{aligned} \dots\dots\dots (12)$$

with the boundary conditions.

$\underline{Y}(0)=\sqrt{1-\alpha}$, $\underline{Y}(1)=0$, $\underline{Y}(2)=y'(0)=a$.

Hence,

$$\begin{aligned} \underline{Y}(3) &= \frac{-\sqrt{1-\alpha}}{3\Gamma(5/2)}, \quad \underline{Y}(4) = \frac{-7\Gamma(3/2)}{2\Gamma(3)\sqrt{\pi}} - \frac{\Gamma(2)}{4\Gamma(3)} a, \\ \underline{Y}(5) &= \frac{\Gamma(2)}{3\Gamma(7/2)} - \frac{\Gamma(2)}{3\Gamma(7/2)} a + \frac{\sqrt{1-\alpha}}{12\Gamma(7/2)} \\ \underline{Y}(6) &= \frac{-2\Gamma(5/2)}{3\sqrt{\pi}\Gamma(4)} + \frac{\sqrt{1-\alpha}}{9\Gamma(4)} + \frac{7\Gamma(3/2)}{8\sqrt{\pi}\Gamma(4)} + \frac{\Gamma(2)}{16\Gamma(4)} a, \\ \underline{Y}(7) &= \frac{-\Gamma(3)}{3\Gamma(9/2)} + \frac{7\Gamma(3/2)}{6\Gamma(9/2)\sqrt{\pi}} + \frac{\Gamma(2)}{6\Gamma(9/2)} a - \\ &\quad - \frac{\Gamma(2)}{12\Gamma(9/2)} - \frac{\sqrt{1-\alpha}}{48\Gamma(9/2)} \end{aligned}$$

Then the approximate solution for \bar{y} up to eight terms is:

$$\bar{y}(x) = \sum_{k=0}^7 \bar{Y}(k) x^{\frac{k}{2}} = \bar{Y}(0)x^0 + \bar{Y}(1)x^{1/2} + \bar{Y}(2)x + \bar{Y}(3)x^{3/2} + \bar{Y}(4)x^2 + \bar{Y}(5)x^{5/2} + \bar{Y}(6)x^3 + \bar{Y}(7)x^{7/2}$$

Hence,

$$\begin{aligned} \bar{y}(x) &= \sqrt{1-\alpha} + ax - \frac{\sqrt{1-\alpha}}{3\Gamma(5/2)} x^{3/2} - \left(\frac{7\Gamma(3/2)}{2\Gamma(3)\sqrt{\pi}} + \frac{\Gamma(2)}{4\Gamma(3)} a \right) x^2 + \\ &\quad + \left(\frac{\Gamma(2)}{3\Gamma(7/2)} - \frac{\Gamma(2)}{3\Gamma(7/2)} a + \frac{\sqrt{1-\alpha}}{12\Gamma(7/2)} \right) x^{5/2} + \left(\frac{-2\Gamma(5/2)}{3\sqrt{\pi}\Gamma(4)} + \frac{\sqrt{1-\alpha}}{9\Gamma(4)} \right. \\ &\quad \left. + \frac{7\Gamma(3/2)}{8\sqrt{\pi}\Gamma(4)} + \frac{\Gamma(2)}{16\Gamma(4)} a \right) x^3 + \left(\frac{-\Gamma(3)}{3\Gamma(9/2)} + \frac{7\Gamma(3/2)}{6\Gamma(9/2)\sqrt{\pi}} \right. \\ &\quad \left. + \frac{\Gamma(2)}{6\Gamma(9/2)} a - \frac{\Gamma(2)}{12\Gamma(9/2)} - \frac{\sqrt{1-\alpha}}{48\Gamma(9/2)} \right) x^{7/2} \end{aligned}$$

and in order to find the constant \bar{a} one can use $\bar{y}(1)=\sqrt{1-\alpha}$ and then we shall find \bar{a} according to the values of α . Following figure (1) represent the upper and lower solution of example one using different values of α .

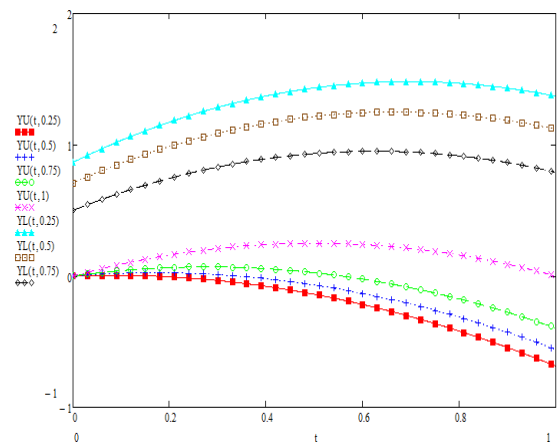


Fig.(1) Upper and Lower Solutions of Example (1) for Different Values of α .

Example (2):

Consider the following nonlinear FFBVP:

$$D_x^{7/4} y(x) + y^2(x) + D_x^{3/4} y(x) = R(x), \quad 0 < x < 1 \tag{13}$$

With boundary conditions:

$$y(0) = 0, \quad y(1) = 0.$$

Where

$$R(x) = \frac{1}{\Gamma(1/4)} x^{-3/4} + x^2 - \frac{1}{\Gamma(5/4)} x^{1/4}. \tag{14}$$

The solution will be of the form $[y, \bar{y}]$.

Now to find \bar{y} we must solve the problem:

$$D_x^{7/4} \bar{y}(x) + \bar{y}^2(x) + D_x^{3/4} \bar{y}(x) = R(x) \tag{15}$$

With boundary conditions:

$$\bar{y}(0) = -\sqrt{1-\alpha} \text{ and } \bar{y}(1) = -\sqrt{1-\alpha} \tag{16}$$

Applying the FDTM for the problems (15) and (16) and according to theorem (5) one can get:

$$\begin{aligned} & \frac{\Gamma\left(\frac{11+k}{4}\right)}{\Gamma\left(\frac{4+k}{4}\right)} \bar{y}(k+7) + \sum_{k_1=0}^k \bar{y}(k_1) \bar{y}(k-k_1) + \\ & + \frac{\Gamma\left(\frac{7+k}{4}\right)}{\Gamma\left(\frac{4+k}{4}\right)} \bar{y}(k+3) = R(k) \end{aligned}$$

or equivalently,

$$\begin{aligned} \bar{y}(k+7) = & \frac{\Gamma\left(\frac{4+k}{4}\right)}{\Gamma\left(\frac{11+k}{4}\right)} \left[- \sum_{k_1=0}^k \bar{y}(k_1) \bar{y}(k-k_1) - \right. \\ & \left. - \frac{\Gamma\left(\frac{7+k}{4}\right)}{\Gamma\left(\frac{4+k}{4}\right)} \bar{y}(k+3) + R(k) \right] \end{aligned} \tag{17}$$

Where

$$R(k) = \frac{1}{\Gamma(1/4)} \delta(k+3) + \delta(k-8) + \frac{1}{\Gamma(5/4)} \delta(k-1) \tag{18}$$

with the boundary conditions:

$$\bar{y}(0) = -\sqrt{1-\alpha}, \quad \bar{y}(1) = 0, \quad \bar{y}(2) = 0, \quad \bar{y}(3) = 0,$$

$$\bar{y}(4) = y'(0) = a,$$

Hence,

$$\bar{y}(5) = 0, \quad \bar{y}(6) = 0, \quad \bar{y}(7) = \frac{(1-\alpha)}{\Gamma(11/4)}, \quad \bar{y}(8) = \frac{1-a}{\Gamma(3)},$$

$$\bar{y}(9) = 0, \quad \bar{y}(10) = 0, \quad \bar{y}(11) = \frac{1}{\Gamma(15/4)} (2a\sqrt{1-\alpha} + 1 - \alpha),$$

$$\bar{y}(12) = \frac{a-1}{\Gamma(4)}, \quad \bar{y}(13) = 0, \quad \bar{y}(14) = \frac{2(\alpha-1)\sqrt{1-\alpha}}{\Gamma(18/4)}.$$

Then the approximate solution for \bar{y} up to 15-terms is:

$$\begin{aligned} \bar{y}(x) = & \sum_{k=0}^{14} \bar{y}(k) x^k = \bar{y}(0) x^0 + \bar{y}(1) x^{1/4} + \bar{y}(2) x^{1/2} \\ & + \bar{y}(3) x^{3/4} + \bar{y}(4) x + \bar{y}(5) x^{5/4} + \bar{y}(6) x^{3/2} + \bar{y}(7) x^{7/4} \\ & + \bar{y}(8) x^2 + \bar{y}(9) x^{9/4} + \bar{y}(10) x^{5/2} + \bar{y}(11) x^{11/4} + \\ & + \bar{y}(12) x^3 + \bar{y}(13) x^{13/4} + \bar{y}(14) x^{14/4}. \end{aligned}$$

and hence,

$$\begin{aligned} \bar{y}(x) = & \sqrt{1-\alpha} + ax + \frac{(\alpha-1)}{\Gamma(11/4)} x^{7/4} + \frac{(1-\alpha)}{\Gamma(3)} x^2 - \\ & \frac{\Gamma(2)}{\Gamma(15/4)} (-2a\sqrt{1-\alpha} + 1 - \alpha) x^{11/4} - \frac{(1-a)}{\Gamma(4)} x^3 + \\ & + \frac{2(\alpha-1)\sqrt{1-\alpha}}{\Gamma(18/4)} x^{14/4}. \end{aligned}$$

and in order to find the constant a one can use $\bar{y}(1) = -\sqrt{1-\alpha}$ we find a for different values of α .

In similar manner one can apply the same approach given for the lower case to find the upper solution as follows:

$$\begin{aligned} & \frac{\Gamma\left(\frac{11+k}{4}\right)}{\Gamma\left(\frac{4+k}{4}\right)} \bar{y}(k+7) + \sum_{k_1=0}^k \bar{y}(k_1) \bar{y}(k-k_1) + \\ & + \frac{\Gamma\left(\frac{7+k}{4}\right)}{\Gamma\left(\frac{4+k}{4}\right)} \bar{y}(k+3) = R(k). \end{aligned}$$

or equivalently,

$$Y(k+7) = \frac{\Gamma\left(\frac{4+k}{4}\right)}{\Gamma\left(\frac{11+k}{4}\right)} \left[\begin{array}{l} -\sum_{k_1=0}^k Y(k_1) Y(k-k_1) \\ \frac{\Gamma\left(\frac{7+k}{4}\right)}{\Gamma\left(\frac{4+k}{4}\right)} Y(k+3) + R(k) \end{array} \right] \quad (19)$$

with boundary conditions:

$$Y(0) = \sqrt{1-\alpha}, \quad Y(1) = 0, \quad Y(2) = 0, \quad Y(3) = 0,$$

$$Y(4) = y'(0) = \bar{a}.$$

Hence,

$$Y(5) = 0, \quad Y(6) = 0, \quad Y(7) = \frac{\alpha-1}{\Gamma(11/4)}, \quad Y(8) = \frac{(1-a)}{\Gamma(3)},$$

$$Y(9) = 0, \quad Y(10) = 0, \quad Y(11) = \frac{\Gamma(2)}{\Gamma(15/4)} (-2\bar{a}\sqrt{1-\alpha} + 1 - \alpha),$$

$$Y(12) = \frac{-(1-a)}{\Gamma(4)}, \quad Y(13) = 0, \quad Y(14) = \frac{-2(\alpha-1)\sqrt{1-\alpha}}{\Gamma(18/4)}.$$

The approximate solution for y up to 15-terms is:

$$\begin{aligned} \bar{y}(x) = & \sqrt{1-\alpha} + \bar{a}x + \frac{(\alpha-1)}{\Gamma(11/4)} x^{7/4} + \frac{(1-a)}{\Gamma(3)} x^2 - \\ & \frac{\Gamma(2)}{\Gamma(15/4)} (-2\bar{a}\sqrt{1-\alpha} + 1 - \alpha) x^{11/4} - \frac{(1-\bar{a})}{\Gamma(4)} x^3 \\ & + \frac{2(\alpha-1)\sqrt{1-\alpha}}{\Gamma(18/4)} x^{14/4}. \end{aligned}$$

and order to find \bar{a} one can use $\bar{y}(1) = -\sqrt{1-\alpha}$ then we shall find \bar{a} for different values of α . Following figure (2) represent the upper and lower solution of example two.

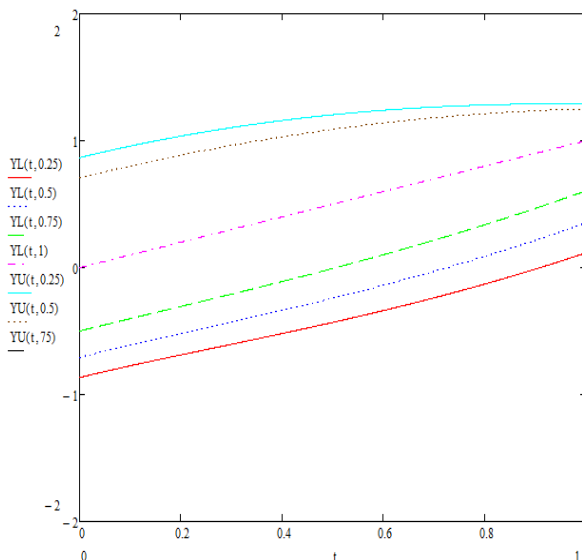


Fig. (2) Upper and Lower Solutions of Example (2) for Different Values of α .

Conclusions

The DTM emphasized our belief that the FDTM is a reliable technique to handle linear and nonlinear fuzzy fractional boundary value problems. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. The crisp solution, i.e, the solution of nonfuzzy fractional differential equations, may be considered as a special case of the fuzzy fractional differential equations with $\alpha=1$.

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الخلاصة

في هذا البحث قدمنا الحل التقريبي لمسائل القيم الحدودية الضبابية ذات الرتب الكسرية بالاعتماد على طريقة تحويل التفاضلات الكسرية والتي استخدمت لحل مسائل القيم الحدودية الضبابية ذات الرتب الكسرية الخطية واللاخطية. الشروط الحدودية في هذه المسائل سوف تكون عبارة عن اعداد ضبابية. الحل لهذا النوع من المسائل سوف يتم حسابه على شكل متسلسلة متقاربة سهلة الحساب. تم حل بعض الامثلة كتوضيح للطريقة. النتائج التي تم الحصول عليها تظهر ان الطريقة سهلة التطبيق وعالية الدقة عند تطبيقها لحل مسائل القيم الحدودية الضبابية ذات الرتب الكسرية.