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Small Prime Modules and Small Prime Submodules

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Abstract

Let R be a commutative ring with identity, and M be a unital (left) R-module. In this paper we introduce and study the concepts: small prime submodules and small prime modules as generalizations of prime submodules and prime modules.

Among the results that we obtain is the following:

An R-module M is small prime if and only if the R-module R/annM is cogenerated by every non-trivial small submodule of M.

1. Introduction

An R-module M is called a prime module if $ann_R M = ann_R N$ for each non-zero submodule N of M, [1]. A proper submodule N of M is called prime submodule if whenever $r\in R$, $x\in M$ and $rx\in N$ implies either $x\in N$ or $r\in[N:M]$, [2]. As a generalization of these concepts, we introduce the concepts of small prime modules and small prime submodules. Where we call M a small prime module if annM = annN for each non-zero small submodule N of M. And we call a submodule N of M small prime submodule if whenever $r\in R$, $x\in M$, (x) is small in M and $rx\in N$ implies either $x\in N$ or $r\in[N:M]$. Where a submodule N of M is called small (notationally, $N \ll M$) if N + K = M for all submodules K of M implies K = M, [3].

The main goal of this research is to study small prime modules and small prime submodules.

This research consists of two sections, in the first section we establish some properties of small prime submodules, and in the second section we give a comprehensive study of small prime modules.

2. Small Prime Submodules

We introduce and study the following concept:

2.1 Definition:

A proper submodule N of an R-module M is called small prime submodule if and only if whenever $r\in R$ and $x\in M$ with $(x)\ll M$ and $rx\in N$ implies either $x\in N$ or $r\in[N:M]$.

A proper ideal I of a ring R is called small prime if I is a small prime submodule of the R-module R. Equivalent a proper ideal I of R is small prime if and only if whenever $r, s\in R$ with $(s)\ll R$ and $rs\in I$ implies either $r\in I$ or $s\in I$.

2.2 Examples and Remarks:

1. Every prime submodule is small prime and the converse is not true in general

For the converse, consider M = Z_{24} as a Z-module and N = $(\bar{6})=\{\bar{0},\bar{6},\overline{12},\overline{18}\}$. N is a small prime submodule of M which is not prime.

To show that N is small prime in M, note that the small submodules of M are: $(\bar{0})$, $(\bar{6})$ and $(\overline{12})$.

$\bar{0}\in N$, $\bar{0}=2\cdot\overline{12}=4\cdot\bar{6}=6\cdot\bar{4}=8\cdot\bar{3}=3\cdot\bar{8}$

(i) $\bar{0}=2\cdot\overline{12}$, $(\overline{12})\ll M$ and $\overline{12}\in N$.

(ii) $\bar{0}=4\cdot\bar{6}$, $(\bar{6})\ll M$ and $\bar{6}\in N$.

(iii) $\bar{0}=6\cdot\bar{4}$, but $(\bar{4})\not\ll M$.

(iv) $\bar{0}=8\cdot\bar{3}$, but $(\bar{3})\not\ll M$.

(v) $\bar{0}=3\cdot\bar{8}$, but $(\bar{8})\not\ll M$.

In the same way we test the elements $\bar{6},\overline{12}$ and $\overline{18}$ of N. So N is small prime in M.

2. If M is an R-module in which every cyclic submodule is small, then every small prime submodule of M is prime.

3. If M is a hollow R-module, then a submodule N of M is small prime iff N is prime.

4. Every proper ideal of the ring Z is small prime.

This follows from the fact that Z is an integral demain and (0) is the only small ideal in Z.

5. (0) is the only small prime submodule of the Z-module Q, since for each $x\in Q$, $(x)\ll Q$, so if N is a non-trivial submodule of Q and $rx \in N$ where $r \in Z$, then it is not necessarily that $x \in N$ or $r \in [N:Q]$.

6. If N is a small prime submodule of M, then it is not necessary that:
(i) $ann_R N$ is a prime ideal of R.
(ii) [N:M] is a prime ideal of R.
For example: Take M = Z_{24} as a Z-module and N= $(\bar{6})$.
Then $ann_Z(\bar{6})$ =4Z is not a prime ideal of Z and [N:M] = 6Z is also not a prime ideal of Z.

7. A submodule of a small prime submodule need not be small prime.
For example: Let N = $(\bar{6})$. N is a small prime submodule of the Z-module Z_{24}. Let K = $(\overline{12}) = \{\bar{0}, \overline{12}\}$. K is not a small prime submodule of N, since $(\bar{6}) \ll Z_{24}$ and $\overline{12} = 2 \cdot \bar{6} \in K$ but $\bar{6} \notin K$ and $2 \notin [K:Z_{24}]$ = 12Z.

8. A direct summand of a small prime submodule is not in general small prime submodule.
For example: Let N = $(\bar{2})$ in the Z-module Z_{24}. N is a small prime submodule of Z_{24} and N = $(\bar{6}) \oplus (\bar{8})$, while $(\bar{8})$ is not a small prime submodule of N, since $(\bar{0})$ and $(\overline{12})$ are the only small submodules of N with $[(\bar{8}):N]$ = 4Z and $\bar{0} = 2 \cdot \overline{12}, \in (\bar{8})$ but $2 \notin [\bar{8}:N]$ and $\overline{12} \notin (\bar{8})$.

9. If $K \subseteq N$ are submodules of M and N is small prime in N, then it is not necessary that K is small prime in N.
For example: Let K = $(\overline{12})$ and N = $(\bar{2})$ in the Z-module Z_{24}. N is small prime in Z_{24} but K is not small prime in N.

10. Let M be an R-module and I be an ideal of R such that $I \subseteq$ annM. Let N be a submodule of M. Then N is small prime R-submodule of M iff N is a small prime R/I submodule of M.

Proof:

Let $\bar{r} \in R/I$, $x \in M$ with $(x) \ll M$ and $\bar{r}x \in N$. But $\bar{r}x$ = rx. Hence the result follows easily.

2.3 Proposition:

Let N and K be small prime submdules of an R-module M such that [N:M]=[K:M]. Then $N \cap K$ is a small prime submodule of M.

Proof:

Let $r \in R$, $x \in M$ with $(x) \ll M$ and $rx \in N \cap K$. Then $rx \in N$ and $rx \in K$. Therefore ($x \in N$ or $r \in [N:M]$) and ($x \in K$ or $r \in [K:M]$). Hence ($x \in N$ and $x \in K$) or $r \in [N:M] = [K:M]$, which implies $x \in N \cap K$ or $r \in [N \cap K:M]$. Therefore $N \cap K$ is small prime submodule of M.

2.4 Proposition:

If N is a small prime submodule of an R-module M and I is any ideal of R, then $[N \underset{M}{:} I]$ is a small prime submodule of M.

Proof:

Put $[N \underset{M}{:} I]$ = K. Let $r \in R$, $x \in M$ with $(x) \ll M$ and $rx \in K$. Then $rxI \subseteq N$. That is $rxa \in N$ $\forall$ $a \in I$. But $(xa) \subseteq (x) \ll M$ implies $(xa) \ll M$, ([4].prop.1.1.3). Therefore either $xa \in N$ $\forall$ $a \in I$ or $r \in [N:M] \subseteq [K:M]$. So, either $xI \subseteq N$ or $r \in [K:M]$. Thus either $x \in K$ or $r \in [K:M]$ which is what we wanted.

2.5 Remark:

The converse of proposition (2.4) is not true in general.
For example: Let M = Z_{24} as a Z-module, I=2Z and N= $(\overline{12})$. Then $[N \underset{M}{:} I] = [(\overline{12}) \underset{Z_{24}}{:} 2Z] = (\bar{6})$ is a small prime submodule of Z_{24}, however N is not a small prime submodule of Z_{24}.

2.6 Proposition:

Let $f: M \longrightarrow M'$ be an R-epimorphism and let N be a small prime submodule of M'. Then $f^{-1}(N)$ is a small prime submodule of M.

Proof:

Let $r \in R$, $x \in M$, $(x) \ll M$ and $rx \in f^{-1}(N)$. Then f(rx) = $rf(x) \in N$. But $(x) \ll M$ implies $(f(x)) \ll M'$, ([4],prop.1.1.3). And N is small prime in M' implies either $f(x) \in N$ or $r \in [N:M']$.
If $f(x) \in N$, then $x \in f^{-1}(N)$
If $r \in [N:M']$, then $rM' \subseteq N$ and hence $rf(M) = f(rM) \subseteq N$ which gives $rM \subseteq f^{-1}(N)$, that is $r \in [f^{-1}(N):M]$. Hence $f^{-1}(N)$ is a small prime submodule of M.

2.7 Remark:

A homomorphic image of a small prime submodule is not in general small prime.

For example: let $f:Z_{24} \longrightarrow Z_{24}$ be such that $f(\bar{x})=2\bar{x}$ $\forall \bar{x} \in Z_{24}$.

The submodule $(\bar{2})$ is small prime in Z_{24}, while $f((\bar{2})) = (\bar{4})$ is not a small prime submodule of Z_{24}.

2.8 Proposition:

Let M be a f.g. faithful multiplication R-module. If N is a small prime submodule of M, then [N:M] is a small prime ideal of R.

Proof:

Let r, $a\in R$ with $(a) \ll R$ and $ra\in[N:M]$. Then $(ra)M\subseteq N$. But $(ra)\subseteq(a) \ll R$ implies that $(ra) \ll R$, ([4],prop.1.1.3) and since M is f.g. faithful multiplication R-module, therefore $(ra)M \ll M$. Hence either $r\in[N:M]$ or $a\in[N:M]$ and hence [N:M] is a small prime ideal of R.

3. Small Prime Modules

In this section, we give and study a generalization of a prime module which is a small prime module. Many properties and characterizations of this concept are obtained. Moreover, we study the relationships between small prime modules and special kinds of modules.

3.1 Definition:

An R-module M is called small prime if and only if annM=annN for each non-zero small submodule N of M.

A ring R is called small prime iff R is a small prime R-module. Equivalentely, R is small prime iff $ann_R I = 0$ for each non-zero small ideal I of R.

3.2 Examples and Remarks:

(1) Every prime R-module is small prime but not conversely in general.
For the converse, Z_6 as a Z-module is small prime but not prime.

(2) Z_4 as a Z-module is not small prime, since $(\bar{2}) \ll Z_4$ and $ann_Z(\bar{2}) = 2Z \neq ann_Z Z_4$.

(3) Every semisimple R-module is small prime but not conversely, since Z as a Z-module is small prime but not semisimple.

(4) Z_{p^∞} and Q/Z as Z-module are not small prime.

(5) Every hollow small prime R-module is prime.

(6) Every integral domain is a small prime ring.

(7) Let M be an R-module and I be an ideal of R such that $I \subseteq annM$. Then M is a small prime R-module iff M is a small prime R/I-module.

(8) Let M_1 amd M_2 be isomorphic R-modules. Then M_1 is small prime iff M_2 is small prime.

Proof (8):

Assume that M_1 is small prime and let $\varphi:M_1 \longrightarrow M_2$ be an R-isomorphism. Let $0 \neq N \ll M_2$. We have to show that $annM_2 = annN$. $\varphi^{-1}(N) \ll M_1$ and $\varphi^{-1}(N) \neq 0$. So, $annM_1 = ann\varphi^{-1}(N)$. But $M_1 \cong M_2$ implies that $annM_1=annM_2$, [5]. And we can show easily that $annN = ann\varphi^{-1}(N)$, which completes the proof.

Now, we give some characterizations of small prime modules.

3.3 Proposition:

An R-module M is small prime iff $annM=ann(m)$ $\forall 0\neq m\in M$ such that $(m) \ll M$.

Proof:

$\Rightarrow$ Is obvious.

$\Leftarrow$ Let $0 \neq N \ll M$ and let $r\in annN$. Then $rm = 0$ $\forall m\in N$ and hence $r\in ann(m)$ $\forall m\in N$. But $(m)\subseteq N \ll M$ implies that $(m) \ll M$, [4]. So, $annM = ann(m)$ (by hyp.). But $annN\subseteq ann(m)$ therefore $annN\subseteq annM$. Thus $annM = annN$ and hence M is small prime.

3.4 Proposition:

An R-module M is small prime iff (0) is a small prime submodule of M.

Proof:

$\Rightarrow$ To prove (0) is a small prime submodule of M. Let $r\in R$, $x\in M$ with $(x) \ll M$ and $rx = 0$.

If $x \neq 0$, then $ann(x) = annM$ (since M is small prime), and hence $r\in annM = [(0):M]$.

If $x = 0$, then $x\in(0)$. Thus (0) is a small prime submodule of M.

$\Leftarrow$ If (0) is a small prime submodule of M.

Let $0 \neq N \ll M$ and let $r\in annN$. Then $rx = 0$ $\forall x\in N$. Therefore $rx\in(0)$. Assume that $x\neq 0$, then $r\in[(0):M] = annM$. Hence $annN \subseteq annM$, therefore annM = annN. That is M is small prime.

3.5 Corollary:

A proper submodule N of an R-module M is small prime submodule iff M/N is a small prime R-module.

3.6 Corollary:

Let M be an R-module. Then the following statement are equivalent:

(1) M is small prime.

(2) annM = ann(m) $\forall 0 \neq m \in M$ and $(m) \ll M$.

(3) (0) is a small prime submodule of M.

The following is another characterization of small prime module.

3.7 Proposition:

An R-module M is small prime iff for each non-zero element $m \in M$ with $(m) \ll M$, there exists an R-isomorphism $f:(m) \longrightarrow \overline{R}$ = R/annM such that f(m) = $\overline{1}$.

Proof:

$\Rightarrow$ Assume that M is a small prime R-module.

Let $0 \neq m \in M$ and $(m) \ll M$. Define $f:(m) \longrightarrow \overline{R}$ such that $f(rm) = r \cdot \overline{1}$ $\forall r \in R$. f is well defined, for if $r_1 m = r_2 m$; $r_1, r_2 \in R$, then $r_1 - r_2 \in ann(m)$. But M is small prime, therefore $r_1 - r_2 \in annM$ (by prop.3.3). Hence $\overline{r_1} = \overline{r_2}$ implies $r_1 \cdot \overline{1} = r_2 \cdot \overline{1}$. It can be easily checked that f is an R-isomorphism. Moreover, for each $r \in R$, $f(rm) = r \cdot \overline{1} = r(1+annM) = r+annM = \overline{r}$ and hence $f(m) = f(1 \cdot m) = \overline{1}$.

$\Leftarrow$ To prove M is small prime

Let $0 \neq m \in M$ and $(m) \ll M$. By hypothesis there exists an R-isomorphism $f:(m) \longrightarrow \overline{R}$ such that f(m) = $\overline{1}$. Let $r \in ann(m)$. Then rm = 0 and hence f(rm) = $\overline{0}$.

Now, $\overline{0}$ = f(rm) = rf(m) = $r \cdot \overline{1}$ = $\overline{r}$. Thus $\overline{r} = \overline{0}$ implies that $r \in annM$. Therefore $ann(m) \subseteq annM$ and hence annM = ann(m). Thus M is small prime (by prop.3.3).

The following are some consequences of prop.(3.7).

3.8 Corollary:

An R-module M is small prime iff every non-trivial cyclic small submodule of M is isomorphic to the R-module R/annM.

3.9 Corollary:

An R-module M is small prime iff all non-trivial cyclic small submodules of M are isomorphic to each other.

3.10 Corollary:

A faithful R-module M is small prime iff every non-trivial cyclic small submodule of M is isomorphic to the R-module R.

It is well-known that: if M is a prime R-module, then annM is a prime ideal of R. This fact does not hold for small prime modules, for instance, the Z-module Z_6 small prime, but $ann_Z Z_6 = 6Z$ is not a prime ideal of Z. However we have the following restricted result:

3.11 Proposition:

If M is a small prime R-module, then annN is a prime ideal of R for each non-trivial small submodule N of M.

Proof:

Let $0 \neq N \ll M$. Let a, b $\in R$ and $ab \in annN$. Then abN = 0. Suppose that $bN \neq 0$. But bN is a submodule of N and $N \ll M$ implies that $bN \ll M$, [4]. Therefore annM = annbN and since $a \in annbN$, then $a \in annM$. On the other hand annM = annN, so $a \in annN$. Therefore annN is a prime ideal of R.

Note that the converse of prop.(3.11) is not true in general.

For example, Z_4 as a Z-module is not small prime while $(\overline{2}) \ll Z_4$ and $ann(\overline{2})$ =2Z is a prime ideal of Z.

3.12 Proposition:

A non-trivial submodule of a small prime R-module is also a small prime R-module.

Proof:

Let M be a small prime R-module and $N \neq 0$ be a submodule of M. Let $0 \neq K \ll N$. Then $K \ll M$, [4]. Therefore annM = annK. But $annM \subseteq annN$. Therefore $annK \subseteq annN$. So, annN = annK and hence N is a small prime R-module.

3.13 Corollary:

f M is an R-module whose injective hull E(M) is small prime, then M is also small prime.

Note that the converse of coro. (3.13) is not true in general, for example Z_2 as a Z-module is small prime and $E(Z_2) = Z_{2^\infty}$ is not small prime Z-module.

3.14 Corollary:

A non-trivial direct summand of a small prime R-module is a small prime R-module.

The converse of prop.(3.14) need not be true in general.

Consider the following example:

Let M = $Z\oplus Z_4$ as a Z-module. M is not small prime, while N = $Z\oplus 0$ is a submodule of M and N is a small prime Z-module.

However the following result holds:

3.15 Proposition:

Let M be an R-module such that RadM is a proper direct summand of M. If RadM is a small prime R-module and annM = annRadM, then M is a small prime R-module.

Proof:

Let $0 \neq m\in M$ and (m)□ M. Then $m\in$RadM, [4]. Hence (m)□ RadM. Therefore annRadM=ann(m) and hence annM=ann(m). So M is small prime (by prop.3.3).

3.16 Proposition:

Let $M = M_1\oplus M_2$ be an R-module such that $annM_1+annM_2 = R$. Then M is small prime R-module iff M_1 and M_2 are small prime R-modules.

Proof:

$\Rightarrow$ To prove M is small prime.

Let $0 \neq N$□ M. Since $annM_1+annM_2 = R$, then $N = N_1\oplus N_2$ where N_i is a submodule of M_i respectively for i=1,2,[6]. But N□ M, therefore N_i□ M_i respectively for i=1,2, [4].

Now, annN = annn($N_1\oplus N_2$) = $annN_1\cap annN_2$ = $annM_1\cap annM_2$ (since M_1 and M_2 are small prime). Therefore annN=ann($M_1\oplus M_2$) = annM. Hence M is small prime.

$\Leftarrow$ Follows by coro.(3.14).

3.17 Proposition:

Let M and M' be two R-modules such that annM = annM'. If $f:M\longrightarrow M'$ is an R-homomorphism and M' is small prime then M is also small prime.

Proof:

Let $0 \neq N$□ M and let $r\in$annN. Then rN = 0. f(rN) = rf(N) = 0 implies $r\in$annf(N). But N□ M gives f(N)□ M', [4]. Therefore $r\in$annM' (since M' is small prime). But annM = annM', so $r\in$annM and hence annN $\subseteq$ annM. Therefore annM = annN which completes the proof.

3.18 Remark:

The condition annM = annM' in prop.(3.17) can not be dropped.

Consider the following example:

Let M = Z_{12} and M' = Z_6 as Z-modules. Define $f:Z_{12}\longrightarrow Z_6$ such that

$f(\overline{x}) = 5\overline{x}\ \ \forall \overline{x}\in Z_{12}$.

Clearly, f is a Z-homomorphism. Z_6 is a small prime Z-module, while Z_{12} is not a small prime Z-module, since $(\overline{6})$□ Z_{12} but $annZ_{12}$ = $12Z \neq 2Z = ann(\overline{6})$.

3.19 Remark:

Epimorphic image of a small prime R-module need not be small prime in general.

Consider the following example:

Let Z and Z_4 be Z-modules and $\pi:Z\longrightarrow Z_4$ be the natural homomorphism. Z is a prime Z-module and hence small prime, while Z_4 is not small prime Z-module.

The following are some consequences of prop.(3.17):

3.20 Corollary:

Let N be a submodule of an R-module M such that annM = [N:M]. If M/N is small prime, then M is also small prime.

3.21 Corollary:

Let N be a small prime submodule of an R-module M such that ann M = [N:M]. Then M is a small prime R-module.

Recall that an R-module M is called coprime if annM = ann(M/N) for every proper submodule N of M, [7].

Z as a Z-module is small prime but not coprime.

Z_{p^∞} as a Z-module is coprime but not small prime.

3.22 Corollary:

Let M be a coprime R-module and N be a submodule of M. If M/N is small prime R-module, then M is a small prime R-module.

3.23 Corollary:

Let M be a coprime R-module and N be a small prime submodule of M. Then M is a small prime R-module.

3.24 Proposition:

Let N be a submodule of an R-module M. If M/N is small prime, then [N:M] = [N:K] for all small submodules K of M containing N.

Proof:

Let $0 \neq K \ll M$ and $K \supseteq N$. Then $K/N \ll M/N$, ([4],prop.1.1.2). But M/N is small prime, so annM/N = annK/N. Hence [N:M]=[N:K].

3.25 Corollary:

If N is a small prime submodule of an R-module M, then [N:M]=[N:K] for all small submodules K of M contaning N.

3.26 Proposition:

If N is a small submodule of an R-module M such that [N:M]=[N:K] for all small submodules K of M containing N, then M/N is small prime.

Proof:

Let $K \neq N$ be a submodule of M containing N such that $K/N \ll M/N$. Then $K \ll M$ ([4,prop.1.1.2). Hence [N:M]=[N:K] and therefore annM/N = annK/N. So M/N is small prime.

3.27 Corollary:

Let N be a small submodule of an R-module M. Then M/N is small prime iff [N:M]=[N:K] for all small submodules K of M containing N.

3.28 Corollary:

Let N be a small submodule of an R-module M. Then N is a small prime submodule of M iff [N:M]=[N:K] for all small submodules K of M containing N.

3.29 Corollary:

Let M be a hollow R-module and N be a submodule of M. Then M/N is small prime iff [N:M]=[N:K] for all submodules K of M containing N.

3.30 Corollary:

Let M be a hollow R-module and N be a submodule of M. Then N is a small prime submodule of M iff [N:M]=[N:K] for all submodules K of M containing N.

It is well-know that : If M is an R-module such that R/annM is an integral domain and M is a torsion-free R/annM-module, then M is a prime R-module, so under these conditions M is a small prime R-module, while the converse is not true in general, for instance Z_6 is a small prime Z-module and Z/6Z is not an integral domain, moreover Z_6 is not a torsion-free Z_6-module. However by using an extra condition we can prove that the converse holds.

3.31 Proposition:

Let M be a small prime R-module in which every non-trivial cyclic submodule is small. Then R/annM is an integral domain and M is a torsion-free R/annM-module.

Proof:

Let $\bar{r}, \bar{s} \in$R/annM. Suppose that $\bar{r}\,\bar{s} = \bar{0}$. Then rs $\in$ annM and hence rs$\in$ann(m) for each $0 \neq m \in M$ (by prop.3.3). So, rsm = 0 implies either r$\in$ann(sm) or s$\in$ann(rm). On the other hand ann(sm) = annM = ann(rm). Therefore either r$\in$annM or s$\in$annM, equivalently, either $\bar{r} = \bar{0}$ or $\bar{s} = \bar{0}$ and hence R/annM is an integral domain.

Now, suppose that $\bar{r}\,m = 0$ with $0 \neq m \in M$ and $\bar{r} \in$R/annM. Then rm = 0 and hence r$\in$ann(m) = annM (by prop.3.3). So, $\bar{r} = \bar{0}$, whence M is a torsion-free R/annM-module.

3.32 Corollary:

Let M be an R-module in which every non-trivial cyclic submodule is small. Then M is small prime R-module iff R/annM is an integral domain and M is a torsion-free R/annM-module.

3.33 Corollary:

Let M be a faithful R-module in which every non-trivial cyclic submodule is small. Then M is small prime iff R is sn integral domain and M is a torsion-free R-module.

3.34 Corollary:

Let M be a hollow R-module. Then M is small prime R-module iff R/annM is an integral domain and M is a torsion-free R/annM-module.

The following result is another characterization of small prime R-module M in terms of the R-module R/annM.

3.35 Theorem:

An R-module M is small prime iff the R-module R/annM is congenerated by every non-trivial small submodule of M.

Proof:

$\Rightarrow$ Suppose that M is small prime.

Let $0 \neq N \ll M$ and let $0 \neq x \in N$. Then $0 \neq (x) \ll M$, [4]. Therefore annM = ann(x) (by prop.3.3).

Now, R/annnM = R/ann(x) $\cong$ Rx = (x) is a submodule of N. Hence there exists a monomorphism from R/annM into N, whence R/annM is cogenerated by N, which proves the "only if" part.

$\Leftarrow$ To prove the "if" part.

Let $0 \neq N \ll M$. Then R/annM is cogenerated by N. So, there exists a monomorphism say, f: R/annM $\longrightarrow N^I$ for some index set I. Let $r \in$ annN. Then rN = 0.

Now, $f(\bar{1}) \in N^I$ and hence $(f(\bar{r}))(i) \in N_i \subseteq N$ $\forall i \in I$. But $f(\bar{r}) = rf(\bar{1})$. Therefore $(f(\bar{r}))(i) = (rf(\bar{1}))(i) = r(f(\bar{1}))(i) \in N$. Hence $(f(\bar{r}))(i) = 0$ $\forall i \in I$ which implies that $f(\bar{r}) = 0$. But f is a monomorphism, therefore $\bar{r} = 0$. Hence $r \in$ annM which is what we wanted.

3.36 Corollary:

A faithful R-module M is small prime iff the R-module R is cogenerated by every non-trivial small submodule of M.

Now, we give the following lemma in order to disccuse the localization of small prime module.

3.37 Lemma:

Let M be an R-module, S be a multiplicatively closed subset of R such that $N_S \neq M_S$ for each proper submodule N of M. Then $N \ll M$ iff $N_S \ll M_S$.

Proof:

$\Rightarrow$ Let $N \ll M$.

Supose that there exists a proper submodule K of M_S such that $N_S + K = M_S$. Then there exists a proper submodule W of M such that $K = W_S$. So, $N_S + W_S = M_S$, implies that $(N + W_S) = M_S$ ([8],Ex.9.11(iii),p.173). Hence N+W=M. But $N \ll M$, therefore W=M and $W_S = M_S$ which is a contradiction.

$\Leftarrow$ Let $N_S \ll M_S$.

Suppose that there exists a proper submodule W of M such that N + W = M. Therefore $(N+W)_S = M_S$. Then $N_S + W_S = M_S$ implies that $W_S = M_S$ which is a contradiction since $W \neq M$.

3.38 Proposition:

Let M be a f.g. R-module. Then M is a small prime R-module iff M_P is a small prime R_P-module for each maximal (prime) ideal P of R.

Proof:

$\Rightarrow$ Suppose that M is a small prime R-module.

Let P be a maximal ideal of R and let $\frac{0}{1} \neq \frac{m}{s} \in M_P$ with $m \in M$ and $s \notin P$. Suppose that $\left(\frac{m}{s}\right) \ll M_P$.Then by lemma 3.37, we get that $(m) \ll M$ and hence ann(m) = annM. But M is f.g. therefore $(\text{ann}(m))_P = (\text{ann}M)_P$, ([9], prop.3.14,p.43). So,ann$\left(\frac{m}{s}\right)$=ann$(m)_P$=annM_P and M_P is a small prime R_P-module.

$\Leftarrow$ Follows similarly.

Recall that, an R-module M is called multiplication if for each submodule N of M there is an ideal I of R such that N = IM, [10].

Next, we study the relationships between small prime modules and multiplication modules.

3.39 Theorem:

Let M be a f.g. faithful multiplication R-module. Then M is a small prime R-module iff R is a small prime ring.

Proof:

$\Rightarrow$ Let I be a non-trivial small ideal of R. Then it can be shown easily that IM is a small submodule of M.

If IM = 0, then $I \subseteq$ annM = 0 which is a contradiction.

Hence $0 \neq IM \ll M$ and since M is small prime, then o = annM = annIM. But annI $\subseteq$ annIM, therefore annI = 0 and hence R is a small prime ring.

$\Leftarrow$ Let $0 \neq N \ll M$. Then $[N:M] \ll R$, ([4],prop.1.1.8). But M being multiplication, implies that N = [N:M]M, [10]. Hence $[N:M] \neq 0$ and since R is a small prime ring, then ann[N:M] = 0.

On the other hand M is faithful yields ann[N:M]M = ann[N:M] and hence annN= 0 which completes the proof.

3.40 Corollary:

Let M be a cyclic faithful R-module. Then M is small prime R-module iff R is a small prime ring.

It is well known that: If M is a multiplication R-module and N is a submodule of M, then N is an R-submodule of M iff N is an S-submodule of M (where $S=End_R(M)$). Hence one can show easily that N is a small R-submodule of M iff N is a small S-submodule of M.

Using this fact we can give the following results:

3.41 Proposition:

Let M be a multiplication R-module. If M is a small prime S-module, then M is a small prime R-module (where $S= End_R(M)$).

Proof:

Let $0\neq N$ be a small R-submodule of M. Suppose that there exists $r\in ann_R N$ and $r\notin ann_R M$. Thus $rM\neq 0$. Define $f:M \longrightarrow M$ by $f(m) = rm\ \forall m\in M$.

Clearly, f is a well-defined R-homomorphism and $f \neq 0$. But $f(N) = rN = 0$ implies $f \in ann_S N = ann_S M$ (since M is a small prime S-module). Hence $f(M) = 0$, that is $f = 0$ which is a contradiction. Thus annN= annM and hence M is a small prime R-module.

Recall that an R-module M is called a scalar module if for all $f \in End_R(M)$; $f \neq 0$ there exists $r\in R$, $r \neq 0$ such that $f(m) = rm\ \forall m \in M$, [11].

3.42 Proposition:

Let M be a f.g. multiplication R-module. If M is a small prime R-module, then M is a small prime S-module.

Proof:

M being a f.g. multiplication R-module implies that M is a scalar R-module, ([11], prop.1.1.10). Let N be a non-trivial small S-submodule of M. Then N is a non-trivial small R-submodule of M. Suppose that there exists $f \in S$, $f \in ann_S N$ and $f\notin ann_S M$ such that $f(x) = rx\ \forall x\in M$. Thus $rN = 0$ and hence $r\in annN= annM$, so, $rM=0$. Hence $f(M) = 0$ which is a contradiction. Therefore $ann_S M = ann_S N$ which completes the proof.

3.43 Corollary:

Let M be a f.g. multiplication R-module. Then M is a small prime R-module iff M is a small prime S-module.

3.44 Proposition:

Let M be a scalar R-module such that annM is a prime ideal of R. Then $S=End_R(M)$ is a small prime ring.

Proof:

annM being a prime ideal of R implies that R/annM is an integral domain. But M is a scalar R-module, implies that $S \cong R/annM$ ([12],lemma 6.1,p.79). Thus S is an integral domain. Hence S is a small prime ring (by Ex. and Rem.3.2(6)).

3.45 Proposition:

Let M be a faithful scalar R-module. Then $S = End_R(M)$ is a small prime ring iff R is a small prime ring.

Proof:

M is a scalar R-module implies that $S \cong R/annM$ ([12],lemma 6.1,p.79). But M being faithful R-module so, $S \cong R$. Therefore R is a small prime iff S is small prime (Ex. and Rem.3.2(8)).

3.46 Proposition:

Let M be a faithful multiplication R-module. Then the following statements are equivalent:

(1) M is a small prime R-module.
(2) R is a small prime ring.
(3) $S = End_R(M)$ is a small prime ring.

Proof:

(1) ⇔ (2) by Th. 3.39.

(2) ⇔ (3) since M is f.g. multiplication R-module then M is a scalar R-module ([11], Coro.1.1.11), so the result follows according to Prop.3.44.

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الخلاصة

لتكن R حلقة ابدالية ذات محايد وليكن M مقاساً أحادياً (أيسر) على الحلقة R. قدمنا ودرسنا في هذا البحث المفاهيم : المقاسات الأولية الجزئية الصغيرة والمقاسات الأولية الصغيرة كأعمال للمقاسات الجزئية الأولية والمقاسات الأولية.

من بين النتائج التي حصلنا عليها:

المقاس M على الحلقة R يكون أولي صغير اذا وفقط اذا كان المقاس R/annM على الحلقة R هو مؤّلد مضاد لكل مقاس جزئي صغير غير تافه من M.