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Numerical Solution of a Uniform Beam Problem Using g-spline-Based Differential Quadrature Method

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Abstract

The differential quadrature method with Chebyshev Gauss Lobatto sampling points is introduced for the modeling of vibration of a uniform beam. The g-spline interpolation function is utilized to obtain the explicit formula of the weighting coefficients for approximation of derivatives. Numerical example is presented to demonstrate the validity and accuracy of the proposed method.

Keywords: Differential Quadrature method, g-spline interpolation formula.

Introduction

Differential Quadrature (DQ) is a numerical method for evaluating derivatives of a sufficiently smooth function, proposed by Bellman and Casti [1]. The basic idea of DQ comes from Gauss Quadrature, which can be considered as a useful method for calculate the integral numerically. Gauss Quadrature is characterized by approximating a definite integral with weighting sum of integrand values at a group of so called Gauss points.

Extension is made for finding the derivatives of various orders of a sufficiently smooth function give rise to DQ [1, 2]. In the other words, the derivatives of a smooth function are approximated with weighting sum of function values at a group of so called nodes [3]. The key procedure in the DQ application lies in the determination of the weighting coefficients. Initially Bellman and his associates proposed two methods to compute the weighting coefficients for the first order derivative. The first method is based on an ill-conditioned algebraic equation system. The second method uses a simple algebraic formulation, but the coordinates of the grid points are fixed by the roots of the shifted Legendre polynomial [4]. In earlier applications of the DQ method, Bellman's first method was usually used because it allows the case of an arbitrary grid point distribution. However, since the algebraic equation system of this method is ill conditioned, the number of the grid points usually used is less than 13; this drawback limits the application of the DQ method [4]. The DQ method and it applications were rapidly developed after the late of 1980, thanks to the innovative work in the computation of the weighting coefficients by many authors see [5, 6, 7, 8, 9]. As a result, the DQ method has emerged as a powerful numerical discretization tool in the past two decades. As compared to the conventional low order finite difference and finite element methods, the DQ can obtain very accurate numerical results using a considerably smaller number of grid points and hence requiring relatively little computational efforts. So for, the DQ method has been efficiently employed in a variety of problems in engineering and physical sciences. A comprehensive review of the differential quadrature method has been given by Bert and Malik [10]. This paper will employ function approximation theory using g-spline interpolation to formulate DQ. Nearly 66 years ago, I.J. Schoenberg [11] introduced the subject of "spline function" since then splines, have proved to be enormously important in various branches of mathematics, such as approximation theory, numerical analysis, numerical treatment of ordinary, integral, partial differential equations, and statistics, etc. There are several type of splines appeared in literature given by [12, 13, 14].

Among these types of spline the so called g-spline interpolation which is necessary to the work of this paper. In 1968 Schoenberg [11] extended the idea of Hermite for splines to specify that the order of derivatives specified may vary from node to node. Schoenberg used the term "g-spline" instead of generalized splines because the natural spline term "generalized spline" describes an extension in a different direction. The g-spline is used to interpolate the HB-data (problem), the data in this problem are the values of the function and it derivatives but without Hermite's condition that the only consecutives be used at each node. Farther, Schoenberge [11] define g-spline as smooth piecewise polynomials, where the smoothness is governed by the incidence matrix, and then proved that g-splines, satisfies what so called the "minimum norm property", which is used for the optimality of the g-spline function defined mathematically by the following inequality:

$$\int_I [f^{(m)}(x)]^2 dx > \int_I [S^{(m)}(x)]^2 dx$$

Where the function S is called a g-spline and it's polynomial spline of degree 2m-1 over the interval I.

The g-spline interpolation

The g-spline interpolation was first presented by Schoenberg [11] as a tool used to specify the interpolatory condition:

$$f^{(j)}(x_i) = y_i^{(j)} \text{ for } (i,j) \in e$$

This condition is the Hermite-Birkhoff problem (and abbreviated by HB-problem), where e is a certain set of ordered pairs which will be defined later in this section.

The HB-Problem, [11]

It is convenient in this subsection to discuss the HB-problem, before; we give the tractable formal definition of the natural g-spline interpolation. Consider the read and distinct node points arranged in ascending order of magnitude and represented as

$$x_1 < x_2 < \cdots < x_K$$

Let α be the maximum of the orders of the derivatives to be specified at the nodes.

Define an incidence matrix E, by:

$$\mathrm{E} = \left[\mathrm{a_{ij}}\right],\ \mathrm{i} = 1,\ 2,\ \ldots,\ \mathrm{k};\ \mathrm{j} = 0,\ 1,\ \ldots, \alpha$$

where:

$$\mathrm{a_{ij}} = \begin{cases} 1, & (i,j) \in e \\ 0, & (i,j) \notin e \end{cases}$$

Here e = {(i, j): i = 1, 2, …, k; j = 0, 1, …, α} has been chosen in such a way that i takes the values 1, 2, …, k; one or more times, while j ∈ {0, 1, …, α} and j = α is attained in at least one element (i, j) of e. Assume also that each row of the incidence matrix E and last column of E should contain some element equals to 1. Let $y_i^{(j)}$ be prescribed real numbers for each (i, j) ∈ e. The HB-problem is to find $f(x) \in C^{\alpha}$, which satisfies the interpolatory condition:

$$f^{(j)}(x_i) = y_i^{(j)} \text{ for } (i,j) \in e \quad \ldots\ldots\ldots\ldots\ldots(1)$$

The matrix E will likewise describes the set of equations (1) if we define the set e by:

$$\mathrm{e} = \{(\mathrm{i},\ \mathrm{j}) \mid \mathrm{a_{ij}} = 1\}$$

then the integer n = $\sum_{\mathrm{i,j}} \mathrm{a_{ij}}$, really is the number of interpolatory conditions required to constitute the system (1).

Definition (1)[11]:

Let m be a natural number, then the HB-problem (1) is said to be m-poised provided that if:

$p(x) \in \Pi_{m-1}$

$p^{(j)}(x_i) = 0$ if (i, j) ∈ e

then:

p(x) = 0.

(Π_{m-1} is the class of polynomials of degree m−1 or less).At this point, the g-spline interpolant of order m to f can be given in terms of the fundamental g-spline functions $L_{ij}(x)$, by:

$$S_m(x) = \sum_{(i,j)\in e} L_{ij}(x) y_i^{(j)} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

where:

$$L_{ij}^{(s)}(x_r) = \begin{cases} 0, & if \ (r,s) \neq (i,j) \\ 1, & if \ (r,s) = (i,j) \end{cases}$$

The definition of g-spline is facilitated by defining a matrix E* which is obtained from the incidence matrix E by adding m − α − 1 columns of zeros to the matrix E. Let $\mathrm{E}^* = [a_{ij}^*]$, where (i = 1, 2, …, k; j = 0, 1, …, m−1), where:

$$a_{ij}^* = \begin{cases} a_{ij}, & if \ j \le \alpha \\ 0, & if \ j = \alpha+1, \alpha+2, \ldots, m-1 \end{cases}$$

If j = α + 1, then E* = E.

Definition (2):

A function S(x) is called natural g-spline for the nodes $x_1, x_2, \ldots, x_k$ and the matrix E* of order m provided that it satisfies the following conditions:

1- $S(x) \in \Pi_{2m-1}$ in (x_i, x_{i+1}), i = 1, 2,., k−1.

2- $S(x) \in \Pi_{m-1}$ in $(-\infty, x_1)$ and in (x_k, ∞).

3- $S(x) \in C^{m-1}(-\infty, \infty)$.

4- If $a_{ij}^* = 0$, then $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$.

Let $\mathcal{S}$ (E*; $x_1, x_2, \ldots, x_k$) denotes the class of all g-spline of order m.

Approximation of Linear Functional with the Sense of g-spline Formula, [11]

Let I = [a, b] be a finite interval containing the knots points $x_1, x_2, \ldots, x_k$ and let us consider a linear functional:

$\mathcal{L}f : C^{\alpha}[a, b] \longrightarrow \mathbb{R}$

of the form:

$$\mathcal{L}f=\sum_{j=0}^{\alpha}\int_{a} a_j(x)f^{(j)}(x)\,dx+\sum_{j=0}^{\alpha}\sum_{i=1}^{n_j} b_{ji}f^{(j)}(x_{ji}) \quad \ldots\ldots(3)$$

where the $a_j(x)$ are piecewise continuous functions in I, $x_{ji} \in$ I and b_{ji} are real constants, we can approximate the functional in equation (3) using the formula:

$$\mathcal{L}f = \sum_{(i,j)\in e} \beta_{ij}f^{(j)}(x_i) + Rf \quad \ldots\ldots (4)$$

Therefore, in order to find the approximation $\mathcal{L}f$ given by (5), which is best in some sense, we propose to determine the real's β_{ij}. I. J. Schoenberg [11] states two procedures to determine β_{ij}. One of them is the so called Sard procedure, which can be summarized by the following theorem:

<u>Theorem:</u>

If $\alpha < m < n$ and the HB-problem (1) is m-poised, then Sard's best approximation (4) to $\mathcal{L}f$ of order m is obtained by operating with $\mathcal{L}$ on both sides of the g-spline interpolation formula (2) of order m.

In other words, the coefficients β_{ij} are given by:

$\beta_{ij} = \mathcal{L}L_{ij}(x)$

where $L_{ij}(x)$ are the fundamental functions of (2).

The g-spline interpolation-based differential Quadrature method

Suppose the function $f(x)$ is sufficiently smooth on the interval $[x_1, x_N]$, and let us consider an m-poised Hermite-Birkoff problem

$f^{(j)}(x_i) = y_i^{(j)},\ (i, j) \in e$

on the N distinct nodes:

$x_1 < x_2 < \ldots < x_N$

Based on differential Quadrature, the first and second order derivatives on each of these nodes are given by:

$$\left.\frac{df}{dx}\right|_{x=x_k} = \sum_{(i,j)\in e} a_{ki}^{(j)} f_i^{(j)},\ k = 1, 2, \ldots, N$$

$$\left.\frac{d^2f}{dx}\right|_{x=x_k} = \sum_{(i,j)\in e} b_{ki}^{(j)} f_i^{(j)},\ k = 1, 2, \ldots, N$$

The coefficients $a_{ki}^{(j)}$ and $b_{ki}^{(j)}$ are the weighting coefficients of the first and second order derivatives the following respectively relation.

Computation of the Weighting Coefficients for the First and Second order Derivatives Using g-spline Interpolation Formula

To find the weights $a_{ki}^{(j)}$ and $b_{ki}^{(j)}$, we need to consider an m-poised Hermite-Birkhoff problem to approximate the function f and our purpose is to construct a polynomial of x , which is of the form

$$f(x) \cong \sum_{(i,j)\in e} L_{ij}(x) f_i^{(j)}$$

Satisfying

$$L_{ij}^{(s)}(x_r) = \begin{cases} 1\ ,\ if\ \ (i, j) = (r, s) \\ 0\ ,\ if\ \ (i, j) \neq (r, s) \end{cases}$$

Then the first and second order derivatives at any grid points can be approximated by the following formulation

$$\left.\frac{df}{dx}\right|_{x=x_k} = \sum_{(i,j)\in e} \left.\frac{dL_{ij}(x)}{dx}\right|_{x=x_k} \cdot f_i^{(j)}$$

and

$$\left.\frac{d^2f}{dx}\right|_{x=x_k} = \sum_{(i,j)\in e} \left.\frac{d^2L_{ij}(x)}{dx^2}\right|_{x=x_k} \cdot f_i^{(j)}$$

Therefore $a_{ki}^{(j)}$ are the coefficients for the first order derivative, obtain by the following formula

$$a_{ki}^{(j)} = \left.\frac{dL_{ij}(x)}{dx}\right|_{x=x_k}$$

and similarly $b_{ki}^{(j)}$ are the coefficients of the second order derivative given by

$$b_{ki}^{(j)} = \left.\frac{d^2L_{ij}(x)}{dx^2}\right|_{x=x_k}$$

In the same manner we may obtain formulae for higher order derivatives by using the higher order weighting coefficients, which are expressed as $e_{ki}^{(j,m)}$ to avoid confusion. They are characterized by the following recurrence formulation.

$$e_{ki}^{(j,m)} = \left. \frac{d^m L_{ij}(x)}{dx^m} \right|_{x=x_k}, (i,j) \in e, k = 1,2,...,N, \quad m = 1,2,3,...,N-1$$

Here we assume that $a_{ki}^{(j)} = e_{ki}^{(j,1)}$ and $b_{ki}^{(j)} = e_{ki}^{(j,2)}$

Free vibration analysis of a uniform beam

The vibration of a uniform beam is governed by a fourth order differential equation. When a numerical method is applied to descretize the spatial derivatives, the ordinary differential equation can be reduced to a set of algebraic equations. The eigenvalue of the resultant algebraic equation system provide the vibrational frequencies of the problem. Usually the, number of interior grid points is equal to the dimension of the resultant algebraic equation system, thus providing the same number of eigenfrequencies. Among all the computed eigenfrequencies, only low frequencies are of practical interest [4].

Governing equations and Boundary conditions

For a beam, three problems are often encountered, i.e., the bending, vibration and column buckling analysis. For a Bernoulli-Euler beam of varying cross-section with length L, the non-dimensional governing equations are

$$s(X)\frac{d^4W}{dx^4} + 2\frac{d\,s(X)}{dx}\frac{d^3W}{dx^3} + \frac{d^2 s(X)}{dx^2}\frac{d^2W}{dx^2} + \frac{L^4 q(x)}{EI_0} = 0 \quad \text{.....(5)}$$

For the bending analysis, and

$$s(X)\frac{d^4W}{dx^4} + 2\frac{d\,s(X)}{dx}\frac{d^3W}{dx^3} + \frac{d^2 s(X)}{dx^2}\frac{d^2W}{dx^2} - \Omega^2 W = 0 \quad \text{.....(6)}$$

For the vibration analysis, and

$$s(X)\frac{d^4W}{dx^4} + 2\frac{d\,s(X)}{dx}\frac{d^3W}{dx^3} + \frac{d^2 s(X)}{dx^2}\frac{d^2W}{dx^2} + \frac{PL^2}{EI_0}\frac{d^2W}{dx^2} = 0 \quad \text{.....(7)}$$

for the column buckling analysis, where $s(X) = EI / EI_0$, $\Omega^2 = \rho A L^4 \omega^2 / EI_0$, $X = x/L$, EI is the beam's flexural rigidity, ρA is the mass per unit length, $q(x)$ is the external distributed load. ω is he dimensional frequency, P is the axial compressive load. For a beam of varying cross-section EI and A are functions of the coordinate x. The governing equation for a beam is $4th$ order ordinary differential equation. For a well-posed problem it requires four boundary conditions. These can be obtained by specifying two boundary conditions at the end $X = 0$, and another two boundary conditions at the end $X = 1$. Basically, there are three types of boundary conditions. For vibration analysis, these boundary conditions are given as

Simply supported end (SS)

W=0 and $\frac{d^2W}{dx^2} = 0$

Clamped end (C)

W=0 and $\frac{dW}{dx} = 0$

Free end (F)

$\frac{d^2W}{dX^2} = 0$ and $\frac{d^3W}{dX^3} = 0$

Numerical Discretization

The selection of locations of the sampling points plays an important role in the accuracy of the solution of the differential equations. Using uniform grids can be considered to be convenient and easy selection method. Quite frequently the DQM delivers more accurate solution using the so called Chebyshev Gauss Lobatto points given by [4],

$$X_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\cdot\pi\right)\right], \; i = 1,2,...,N \quad \text{....(8)}$$

With the coordinates of mesh points given by (8), the g-spline-based differential quadrature weighting coefficients can be easily computed. These weighting coefficients can then be used to discetize equations (5), (6) and (7). Using the DQ method equations (5), (6) and (7) can be discretized as

$$s^{(2)}(X_i)\sum_{(i,j)\in e} e_{ki}^{(j,2)}W_i^{(j)} + 2s^{(1)}(X_i)\sum_{(i,j)\in e} e_{ki}^{(j,3)}W_i^{(j)} + s(X_i)\sum_{(i,j)\in e} e_{ki}^{(j,4)}W_i^{(j)} = -\frac{L^4}{EI_0}q(X_i),(i,j)\in e \text{ ....(9)}$$

$$s^{(2)}(X_i)\sum_{(i,j)\in e} e_{ki}^{(j,2)}W_i^{(j)} + 2s^{(1)}(X_i)\sum_{(i,j)\in e} e_{ki}^{(j,3)}W_i^{(j)} + s(X_i)\sum_{(i,j)\in e} e_{ki}^{(j,4)}W_i^{(j)} = \Omega^2 W_i^{(j)},(i,j)\in e \text{ ...(10)}$$

$$s^{(2)}(X_i)\sum_{(i,j)\in e} e_{ki}^{(j,2)}W_i^{(j)} + 2s^{(1)}(X_i)\sum_{(i,j)\in e} e_{ki}^{(j,3)}W_i^{(j)} + s(X_i)\sum_{(i,j)\in e} e_{ki}^{(j,4)}W_i^{(j)} = -\frac{PL^4}{EI_0}\sum_{(i,j)\in e} e_{ki}^{(j,2)}W_i^{(j)},(i,j)\in e$$

where $w_i^{(j)}$ are the functional value and its derivative at the grid point x_i but without Hermite's condition, $S^{(2)}(x_i)$ and $S^{(1)}(x_i)$ are respectively the first and second order derivatives of $S(x)$ at x_i, $e_{ki}^{(j,m)}, m = 2,3,4,...,$. are the DQ weighting coefficients of the mth order derivative.

With proper implementation of the boundary conditions, equation (9)we formulate matrix as

$$[A_d]\{W_i^{(j)}\} = \{b\} \text{ ....(11)}$$

Where $[A_d]$ is a matrix, $\{W_i^{(j)}\}$ is a vector of unknowns and $\{b\}$ is known vector.

The algebraic equation (11) can be solved using the direct methods such as the LU decomposion, or iterative methods such as the SOR approach.

In similar manner, Equation (10), can be written as

$$[A_v]\{W_i^{(j)}\} = \Omega^2\{W_i^{(j)}\} \text{...............................(12)}$$

and equation (25) can be put to

$$[A_b]\{W_i^{(j)}\} = \frac{PL^2}{EI_0}[B]\{W_i^{(j)}\} \text{......................(13)}$$

Where $[A_v],[A_b]$ and $[B]$ are matricies, Equation (12) and (13) are eigenvalue systems. The frequency of a free vibration problem can be obtained from the eigenvalues of equation (12). The eigenvalues of equation (12) and (13) can be calculated using the standard solvers such as the QR algorithm.

Direct substitution of Boundary Conditions into Discrete Governing Equations

To implement the simply supported, clamped conditions and their combinations. The essence of the approach is that the Dirichlet condition is implemented at the boundary point, while the derivative condition is discretized by DQ method. The discretizeditive conditions at the two ends are then combined to given derivative conditions at the two ends then combined to give the solutions $W_2^{(0)}$ and $W_{N-1}^{(0)}$. The expressions of $W_2^{(0)}$ and $W_{N-1}^{(0)}$ are then substituted into the discrete governing equation which is applied to the interior order pairs $(i,j)\in e/\{(2,0),(N-1,0)\}$.

For any combination of the clamped and simply supported conditions at the two ends, the discrete boundary conditions using the DQ method can be written as

$$W_1^{(0)} = 0 \text{..}(14a)$$

$$\sum_{(i,j)\in e} e_{1i}^{(j,n0)}W_i^{(j)} = 0 \text{......................}(14b)$$

$$W_N^{(0)} = 0 \text{...}(14c)$$

$$\sum_{(i,j)\in e} e_{Ni}^{(j,n1)}W_i^{(j)} = 0 \text{......................}(14d)$$

Where $n0$ and $n1$ may be taken as 1 or 2.

We shall treat only the following two sets of boundary conditions

$$\left.\begin{array}{l} n0=1, n1=2 \text{ ...clamped} - \text{simply supported} \\ n0=2, n1=2 \text{...simply supported} - \text{ simply supported} \end{array}\right\} \text{.................(15)}$$

Equation (14a) and (14c) can be easily substituted into (14) and it's clear this is not the case for equations (14b) and (14d).

However one can couple these two equations together to give the solutions $W_2^{(0)}$ and $W_{N-1}^{(0)}$ as

$$W_2^{(0)} = \frac{1}{AXN}\cdot\sum_{(i,j)\in e^*} AXK1\cdot W_i^{(j)} \text{ ....}(16a)$$

$$W_{N-1}^{(0)} = \frac{1}{AXN}\cdot\sum_{(i,j)\in e^*} AXKN\cdot W_i^{(j)} \text{ ....}(16b)$$

Where

$e^* = e / \{(2,0),(N-1,0)\}$

$AXK1 = e_{1,i}^{(j,n0)} \cdot e_{N,N-1}^{(j,n1)} - e_{1,N-1}^{(j,n0)} \cdot e_{N,i}^{(j,n1)}$

$AXKN = e_{1,2}^{(j,n0)} \cdot e_{N,i}^{(j,n1)} - e_{1,i}^{(j,n0)} \cdot e_{N,2}^{(j,n1)}$

$AXN = e_{N,2}^{(j,n1)} \cdot e_{1,N-1}^{(j,n0)} - e_{1,2}^{(j,n0)} \cdot e_{N,N-1}^{(j,n1)}$

According to equation (16a) and (16b) $W_2^{(0)}$ and $W_{N-1}^{(0)}$ are expressed in terms of $W_i^{(j)}$, $(i,j) \in e^*$ and can be easily substituted into equation (10) . In order to find the values of $W_i^{(j)}, (i,j) \in e^*$ the discretized governening equation (10) has to applied at interior points $(i,j) \in e^*$. Substituting equations (14b), (14c), (16a) and (16b) into equation (10) gives

$$s^{(2)}(X_i) \sum_{(i,j)\in e^*} C_1 W_i^{(j)} + 2s^{(1)}(X_i) \sum_{(i,j)\in e^*} C_2 W_i^{(j)}$$

$$+ s(X_i) \sum_{(i,j)\in e^*} C_3 W_i^{(j)} = \Omega^2 W_i^{(j)} \; ,(i,j) \in e^* \; ....(17)$$

where

$$C_1 = e_{k,i}^{(j,2)} - \frac{e_{k,2}^{(j,2)} \cdot AXK1 + e_{k,N-1}^{(j,2)} \cdot AXKN}{AXN}$$

$$C_2 = e_{k,i}^{(j,3)} - \frac{e_{k,2}^{(j,3)} \cdot AXK1 + e_{k,N-1}^{(j,3)} \cdot AXKN}{AXN}$$

$$C_3 = e_{k,i}^{(4)} - \frac{e_{k,2}^{(j,4)} \cdot AXK1 + e_{k,N-1}^{(j,4)} \cdot AXKN}{AXN}$$

It is noted that the equation (17) has (N-4) equations with (N-4) unknowns which can be written in matrix form written as

$[A]\{W_i^{(j)}\} = \Omega^2 \{W_i^{(j)}\}$

where Ω^2 represent the eigenvalues of the above system

Numerical Example: Free Vibration Analysis of a uniform Beam Using g-Spline Interpolation-Based Differential Quarature

In this section the free vibration analysis of a uniform beam as given in equation (10) with $(s(x)=1)$ will be treated.

Two combinations of boundary conditions given by (15) are considered. Applying the approach given in section (5) by considered two different sets of Hermite-Birkhoff problems in the first one the usual differential quadrature occurs if we assume that $N = K$ and $e = \{(i,0), i = 1,...,k-1\}$ since the usual differential quadrature based on lagrange interpolating polynomial and the Hermit-Birkhoff problem can be reduced to a lagrange problem if we consider $j = 0$ in each order pairs of e. While the second set represent that our approach can be considered as a generalization to the usual differential quadratrue and this can be illustrated by the following cases.

Case1:-

To construct the approximate solution via g-spline-based differential quadrature method an m-poised Hermite-Birkhoff problem must chosen.

In this case we shall take a 5-poised Hermite-Birkhoff problem with

$e_1 = \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0)\}$

we shall seek

$S_5(x) \in S(E^*, x_1, x_2, x_3, x_4, x_5, x_6)$ where

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

and for which

$S_5(x_i) = W_i^{(j)}, \; (i,j) \in e_1$

Applying the g-spline based differential quadrature method for the governing equation (10) $(S(x)=1)$ and the boundary conditions given by (15) using the node points (8) with $N = 6$, hence we have an eigenvalue system given by (12).

Following Table (1) represent a comparison of the low natural frequency (Ω) of a uniform beam using g-spline-based differential quadrature with the exact solution given by Belvins [15].

Table (1)

Comparison of natural frequency (Ω) of a uniform beam using g-spline interpolation-based differential quadrature using e₁.

Boundary Conditions	Ω (DQ)	Ω (Exact)	Error%
SS-SS	9.8669	9.8696	–0.0027
C-SS	15.4682	15.4182	0.05

It is noted that all calculations are performed by a computer programs written by MATLAB14. The fundamental g-spline functions $L_{10}(x), L_{20}(x), L_{30}(x), L_{40}(x), L_{50}(x)$ and $L_{60}(x)$ are given in appendix A.

Case 2:-

In this case we shall take a 5-poised Hermite-Birkhoff problem with $e_2 = \{(1,0),(2,0),(3,0),(4,1),(5,1),(6,0)\}$ we shall seek $S_5(x) \in S(E^*, x_1, x_2, x_3, x_4, x_5, x_6)$ where

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and for which

$S_5^{(j)}(x_i) = W_i^{(j)}, \ (i, j) \in e_2$

Similarly applying the g-spline based differential quadrature method for the governing equation (10) $(S(x) = 1)$ and the boundary conditions given by (15) using the node points (8) with $N = 6$, hence we have an eigenvalue system given by (12).

Following Table (2) represent a comparison of the low natural frequency (Ω) of a uniform beam using g-spline-based differential quadrature with the exact solution given by Belvins [15].

Table (2)
Comparison of natural frequency (Ω) of a uniform beam using g-spline interpolation-based differential quadrature using e_2.

Boundary Conditions	Ω ***(DQ)***	Ω ***(Exact)***	***Error%***
SS-SS	9.5773	9.8696	-0.1923
C-SS	15.7376	15.4182	0.3194

The fundamental g-spline functions $L_{10}(x), L_{20}(x), L_{30}(x), L_{41}(x), L_{51}(x)$ and $L_{60}(x)$ are given in appendix B.

Conclusion

It is clear that the g-spline-based differential quadrature can be considered as a generalization to the usual differential quadrature method as it is pointed in section (6). Also, from Table (1) and Table (2) one can conclude that g-spline based differential quadrature given reasonable results although we used a small number of node points.

Appendix A:

$$L_{10} = 1 - 16.713x + 78.366x^2 - 143.193x^3 + 93.468x^4 - 55.423(x-0)_+^9 + 110.86(x-.0955)_+^9 - 110.985(x-.3456)_+^9 + 111.195(x-.6546)_+^9 - 111.41(x-.9046)_+^9 + 55.763(x-1)_+^9$$

$$L_{20} = 20.379x - 127.056x^2 + 257.962x^3 - 177.142x^4 + 177.142(x-0)_+^9 - 221.75(x-.0955)_+^9 + 221.999(x-.3456)_+^9 - 222.419(x-.6546)_+^9 + 222.85(x-.9046)_+^9 - 111.541(x-1)_+^9$$

$$L_{30} = -5.24x + 72.038x^2 - 192.538x^3 + 151.627x^4 - 110.985(x-0)_+^9 + 221.999(x-.0955)_+^9 - 222.249(x-.3456)_+^9 + 222.669(x-.6546)_+^9 - 223.101(x-.9046)_+^9 + 111.666(x-1)_+^9$$

$$L_{40} = 2.519x - 37.745x^2 + 129.453x^3 - 120.173x^4 + 111.195(x-0)_+^9 - 222.419(x-.0955)_+^9 + 222.669(x-.3456)_+^9 - 223.091(x-.6546)_+^9 + 223.523(x-.9046)_+^9 - 111.877(x-1)_+^9$$

$$L_{50} = -1.681x + 25.66x^2 - 92.759x^3 + 94.766x^4 - 111.41(x-0)_+^9 + 222.85(x-.0955)_+^9 - 223.101(x-.3456)_+^9 + 223.523(x-.6546)_+^9 - 223.956(x-.9046)_+^9 + 112.094(x-1)_+^9$$

$$L_{60} = .737x - 11.273x^2 + 41.076x^3 - 42.545x^4 + 55.763(x-0)_+^9 - 111.541(x-.0955)_+^9 + 111.666(x-.3456)_+^9 - 111.877(x-.6546)_+^9 + 112.094(x-.9046)_+^9 - 56.105(x-1)_+^9$$

Appendix B

$$L_{10} = 1 - 17.597x + 90.409x^2 - 174.268x^3 + 116.155x^4 - 42.812(x-0)_+^9 + 68.302(x-0.0955)_+^9 - 25.613(x-0.3458)_+^9 - 26.239(x-0.6546)_+^8 + 6.394(x-0.9046)_+^8 + 0.122(x-1)_+^9$$

$$L_{20} = 22.056x - 150.009x^2 + 318.05x^3 - 221.722x^4 + 86.24(x-0)_+^9 - 137.588(x-0.0955)_+^9 + 51.595(x-0.3458)_+^9 + 52.856(x-0.6546)_+^8 - 12.881(x-0.9046)_+^8 - 2.221(x-1)_+^9$$

$$L_{30} = -7.198x + 99.57x^2 - 271.589x^3 + 216.346x^4 - 101.248(x-0)_+^9 + 161.532(x-0.0955)_+^9 - 60.573(x-0.3458)_+^9 - 62.054(x-0.6546)_+^8 + 15.123(x-0.9046)_+^8 + 0.29(x-1)_+^9$$

$$L_{41} = -0.752x + 11.049x^2 - 36.161x^3 + 32.963x^4 - 19.359(x-0)_+^9 + 30.886(x-0.0955)_+^9 - 11.582(x-0.3458)_+^9 - 11.865(x-0.6546)_+^8 + 2.892(x-0.9046)_+^8 + 0.055(x-1)_+^9$$

$$L_{51} = -0.606x + 8.835x^2 - 28.158x^3 + 24.221x^4 - 11.705(x-0)_+^9 + 18.674(x-0.0955)_+^9 - 7.002(x-0.3458)_+^9 - 7.174(x-0.6546)_+^8 + 1.748(x-0.9046)_+^8 + 0.033(x-1)_+^9$$

$$L_{60} = 2.739x - 39.97x^2 + 127.807x^3 - 110.779x^4 + 57.82(x-0)_+^9 - 92.246(x-0.0955)_+^9 + 34.591(x-0.3458)_+^9 + 35.437(x-0.6546)_+^8 - 8.636(x-0.9046)_+^8 - 0.165(x-1)_+^9$$

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الخلاصة

ان طريقة التفاضلات التربيعيه مع نقاط جيبيجف– كاوس – لوبانو قد استخدمت لتمثيل النموذج الرياضي لاهتزاز عامود منتظم. استخدمت دوال السبلاين **g-** للاندراج للحصول على صيغة لمعاملات الاوزان لتقريب المشتقات. ثم تقديم مثال عددي يوضح فعالية ودقة الطريقه المقترحه.