# PreCartan G-Space

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**Abstract**

In this paper a preCartan G-space is our aim. Now we list the following some results that we have gotten: **(i)** a Cartan G-space is preCartan. **(ii)** We introduced some results on a net with a preopen set. **(iii)**We introduced this space (preCartan G-space) and give enough examples and theorems about it, where we study its properties, subspace, product, and the equivarianthomeomorphic image.

Keywords: Preopen, preclosed, preneighborhood, precluster, preconvergence, strongly preopen function, preCartan G-space.

## Introduction

The first step of studying preopen set was done in 1984 [5]. The authors were defined a set A to be preopen if $A \subseteq \overline{A}^{o}$ and that the intersection of an open set and a preopen set is preopen.

The set of all preopen sets of a topological space X is denoted by PO(X), the complement of a preopen set is called preclosed [5]. The intersection of all preclosed sets containing A is called the preclosure of A, denoted by $\overline{A}^{p}$, which is the smallest preclosed set containing A [5], [11]. Preneighborhood is introduced in [7]. Preopen function is introduced in [4], where strongly preopen function is introduced in [6]. By occasion, the definitions of a precluster point of a net and a preconvergence net could be found in [8], [13]. The aim of this paper is to introduce another type of a Cartan G-space which we call apreCartan G-space.On the other hand, a Cartan G-space is introduced by Palais in [2].The space in the sense of Palais is assumed to be a completely regular and a Hausdorff while G is a locally compact.

## Preliminaries:

In this section, we recall the following theorems that we need:

***Theorem 2.1 [3]:***

**(i)** A topological space X is $T_2$ if and only if every convergent net in X has a unique limit.

**(ii)** A topological space X is compact if and only if each net in X has a cluster point.

**(iii)** A net has y as cluster point if and only if it has a subnet which converges to y.

***Theorem 2.2 [3]:***

Let f be a function from a topological space X in to a topological space Y.
Then f is continuous at $x \in X$ if and only if whenever $x_\alpha \to x$ in X, then $f(x_\alpha) \to f(x)$

***Theorem 2.3 [9]:***

Let X be a topological space and Y $\subset$X. Then Y is open if and only if no net in X-Y can converge in a point in Y.

***Theorem 2.4 [12]:***

For each $x \in X$, the isotropy subgroup $G_x$ at $x$ is closed.

***Theorem 2.5 [6]:***

Let $(X_i)_{i \in I}$ be a family of topological spaces and $\emptyset \neq A_i \subseteq X_i$ for each $i \in I$. Then $\prod_{i\epsilon I} A_i$ is preopen in $\prod_{i\epsilon I} X_i$ if and only if $A_i$ is preopen in $X_i$ for each $i \in I$ and $A_i$ is a non dense for only finitely many $i \in I$.

***Theorem 2.6 [6]:***

If Uis a preopen subspace of a topological space X, and V isa preopen subset of $(U, \tau/U)$, then $V$ is preopen in $X$.

***Theorem 2.7 [10]:***

A subset A of a topological space X is preclosed set if and only if A= $\overline{A}^{p}$.

***Theorem 2.8 [13]:***

Let X be a topological space and A $\subset$X, $x \in X$. Then $x \in \overline{A}^{p}$ if and only if there is a net $(x_\alpha)_{\alpha \in \Lambda}$ in A such that $x_\alpha \overset{p}{\propto} x$.

## PreCartan G-space:

A new G-space is introduced in this section which we call a preCartan G-space, which is weaker than a Cartan G-space. But first we state and prove the following theorem.

***Theorem 3.1:***

Let $(x_\alpha)_{\alpha\in\Lambda}$ be a net ina topological space X such that $x_\alpha \overset{p}{\propto} x, x \in X$ and let $A \in PO(X)$ such that $x \in A$. Then there exists a subnet $(x_{\alpha_\mu})$ in A of the net $(x_\alpha)$ such that $x_{\alpha_\mu} \to x$.

***Proof:***

LetU be an open subset of X. Then $U \cap A$ is a preopen set such that $x \in U \cap A$. $(x_\alpha)_{\alpha\in\Lambda}$ is frequently in $U \cap A$. Let

$M = \{(\alpha, U \cap A) | \alpha\in\Lambda, U \text{ is an open subset of } X, x \in U, and\ x_\alpha \in U \cap A\}$.

Suppose that M be ordered as follows:

$(\alpha_1, U_1 \cap A) \le (\alpha_2, U_2 \cap A)$ if and only if $\alpha_1 \le \alpha_2$ and $U_1 \subseteq U_2$.

Clear that $\le$ is reflexive and transitive relations.

At the present time, let $(\alpha_1, U_1 \cap A)$ and $(\alpha_2, U_2 \cap A)$ be in M.

$(U_1 \cap U_2) \cap A \in PO(X)$ and $x\in (U_1\cap U_2) \cap A$

So $(x_\alpha)$ is frequently in $(U_1 \cap U_2) \cap A$.

Since $\Lambda$ is a directed set and $\alpha_1, \alpha_2\in\Lambda$, then there exists $\alpha_3'\in\Lambda$ such that $\alpha_1 \le \alpha_3'$ and $\alpha_2 \le \alpha_3'$.

Therefore, there exists $\alpha_3\in\Lambda$ such that $x_{\alpha_3}\in(U_1 \cap U_2) \cap A$ and $\alpha_3' \le \alpha_3$.

i.e. $(\alpha_3, (U_1 \cap U_2) \cap A) \in M$ such that $\alpha_1\le\alpha_3$, $\alpha_2\le\alpha_3$ and $U_1 \cap U_2\subseteq U_1$, $U_1\cap U_2\subseteq U_2$.

Hence $(\alpha_1, U_1 \cap A) \le (\alpha_3, (U_1 \cap U_2) \cap A)$ and $(\alpha_2, U_2 \cap A) \le (\alpha_3, (U_1 \cap U_2) \cap A)$

So M is a directed set.

Define $g:M\to\Lambda$ such that $g(\alpha, U\cap A) = \alpha$. To prove that *xog* satisfying a subnet conditions.

Let $(\alpha_1, U_1 \cap A) \le (\alpha_2, U_2 \cap A)$. Then $\alpha_1 \le \alpha_2$ i.e. $g(\alpha_1, U_1 \cap A) \le g(\alpha_2, U_2 \cap A)$.

Let $\alpha\in\Lambda$.

On the other hand, since $X \cap A = A$ is a preopen subset of X which contains *x*, then there exists $\alpha'\in\Lambda$ such that $x_\alpha\in X \cap A$ and $\alpha \le \alpha'$.

So $(\alpha', X \cap A)\in M$, such that:

$\alpha \le \alpha' = g(\alpha', X \cap A)$

Hence g defines a subnet of the net $(x_\alpha)$.

Now, let $U_o$ be any open subset of X which contains *x*.

Then $U_o \cap A$ is a preopen subset of X which contains *x*.

We could find $\alpha_o\in\Lambda$ such that $x_{\alpha_o}\in U_o \cap A$.

So $(\alpha_o, U_o \cap A) \in M$

Hence for each $(\alpha, U \cap A) \in M$ and $(\alpha_o, U_o\cap A) \le (\alpha, U \cap A)$, we have $\alpha_o \le \alpha$ and $U \subseteq U_o$.

So $x_\alpha\in U \subseteq U_o$.

This subnet is eventually in every neighborhood which contains *x*.

Hence it is converges to $x\in A$.

***Definition 3.2:***

A G-space X is called a preCartan G-space if every point of X has a thin preneighborhood.

***Example 3.3:***

(i) (R, + ) with the usual topology is a locally compact topological group, and the set:

$D=\{(x, y)\in R^2 \setminus\{(0, 0)\} \mid x\ge 0, y\ge 0\}$ with the relative usual topology is a completely regular $T_2$ space.Let R acts on D as follows:

$\pi:R\times D\to D$ such that $\pi(t, (x, y))=(xe^{-t}, ye^t)$ for each $t\in R$, $(x, y)\in D$. Clear that D is an R-space.

To show that D is a preCartan R-space.

Let $(x, y)\in D$ and $U = (x-\varepsilon, x+\varepsilon)$ be a preneighborhood of x in

$L= \{(x, 0)\in R^2\setminus\{(0, 0)\} \mid x \ge 0\}$,

where ε does not equal neither x nor –x. Let:

$W=\{(0, y)\in R^2\setminus\{(0, 0)\} \mid y \ge 0\}$.

By theorem 2.5 we get that $U\times W$ is a preneighborhood of (x, y) in D.

Before we prove $((U\times W, U\times W))=((U, U))$, we need to show that W is an R-space and then we can continue solving the example.

(W, +) with the relative usual topology is a topological group which is locally compact but not compact and R with the usual topology is a completely regular $T_2$ space. Then R acts on W as follows:

$\pi_1: R\times W\to W$, such that $\pi_1(t, y) = ye^t$ for each $y\in W$, $t \in R$. Clear that W is an R-space. Now to prove $((U\times W, U\times W)) = ((U, U))$.

$g\in ((U, U)) \leftrightarrow gU\cap U \ne\varnothing \leftrightarrow (gU\cap U)\times W \ne\varnothing\times W \leftrightarrow gU\times W \cap U\times W \ne\varnothing \leftrightarrow$ since by [8] W is invariant $gU\times gW\cap U\times W\ne\varnothing \leftrightarrow g\in ((U\times W, U\times W))$.

Hence $((U\times W, U\times W)) = ((U, U))$.

Yet we have to show that ((U, U)) has a compact closure.

$e^{-t_1}(x-\varepsilon) = x+\varepsilon \Rightarrow t_1 = \ln((x-\varepsilon)/(x+\varepsilon))$

$e^{-t_2}(x+\varepsilon) = x-\varepsilon \Rightarrow t_2 = \ln(((x+\varepsilon)/(x-\varepsilon))$

If $x > 0$, then $t_1 = -t_2$ and the set:

$((U, U)) = \{g \in G \mid gU \cap U \neq \varnothing\} = (-t_2, t_2)$ has a compact closure.

Hence D is a preCartan R-space.

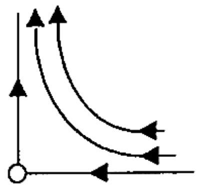

*Fig. (I).*

(ii) $(R\backslash\{0\}, \cdot)$ with the usual topology is a locally compact non-compact topological group. Besides, $R^2$ with the usual topology is a completely regular Hausdorff space.

Then $R\backslash\{0\}$ acts on $R^2$ as follows:

$\pi: R\backslash\{0\} \times R^2 \rightarrow R^2$

is defined by:

$\pi(r, (x, y)) = (rx, ry)$

for each $r \in R\backslash\{0\}$ and $(x, y) \in R^2$.

Clear that $R^2$ is $R\backslash\{0\}$ -space.

But $R^2$ is not preCartan $R\backslash\{0\}$-space, since $(0, 0) \in R^2$ has no thin preneighborhood since for any preneighborhood U of (0, 0) the set $((U, U)) = R\backslash\{0\}$ is not relatively compact in $R\backslash\{0\}$.

***Proposition 3.4:***

A Cartan G-space is preCartan.

***Proof:***

Clear.

***Proposition 3.5:***

If X is a preCartan G-space, then:

**a)** Each orbit of X is preclosed.

**b)** For each $x \in X$ the isotropy subgroup $G_x$ at x is compact.

***Proof:a)***

Let X be a preCartan G-space to prove that Gx is preclosed in X (i.e., $Gx = \overline{Gx}^p$), we have to show that $\overline{Gx}^p \subseteq Gx$.

Let $y \in \overline{Gx}^p$. Then by 2.8, there is a net $(g_\alpha x)$ in Gx such that $g_\alpha x \overset{p}{\propto} y$.

Since X is preCartan, then there exists U a thin preneighborhood of y.

By 3.1, there is a subnet $(g_{\alpha_\mu} x)$ of the net $(g_\alpha x)$ in U such that $g_{\alpha_\mu} x \rightarrow y$.

Fixing $\alpha_o$, then $(g_{\alpha_\mu} g_{\alpha_o}^{-1})(g_{\alpha_o} x) = g_{\alpha_\mu} x$.

To prove $g_{\alpha_\mu} g_{\alpha_o}^{-1} \in ((U, U))$.

Because $(g_{\alpha_\mu} x)$ is in U, then so is $(g_{\alpha_o} x)$.

Hence $(g_{\alpha_\mu} g_{\alpha_o}^{-1})(g_{\alpha_o} x)$ lies in $(g_{\alpha_\mu} g_{\alpha_o}^{-1})$ U.

i.e., $U \cap (g_{\alpha_\mu} g_{\alpha_o}^{-1}) U \neq \varnothing$.

Then $g_{\alpha_\mu} g_{\alpha_o}^{-1} \in ((U, U))$.

Since ((U, U)) is relatively compact, then by 2.1(ii), $(g_{\alpha_\mu} g_{\alpha_o}^{-1})$ has a cluster point say $g \in G$.

Hence by 2.1(iii), we get that $(g_{\alpha_\mu} g_{\alpha_o}^{-1})$ has a subnet which converges to g.

So $g_{\alpha_\mu} g_{\alpha_o}^{-1} \rightarrow g$, then $g_{\alpha_\mu} \rightarrow g g_{\alpha_o}$.

By 2.2, we get that $g_{\alpha_\mu} x \rightarrow g\, g_{\alpha_o} x$.

Since X is $T_2$, then by 2.1(i) we have $y = g\, g_{\alpha_o} x \in Gx$.

Hence $\overline{Gx}^p \subseteq Gx$.

But we have $Gx \subseteq \overline{Gx}^p$.

Therefore $Gx = \overline{Gx}^p$. So by 2.7 we get that Gx is preclosed in X.

**b)** Let $x \in X$.

Since X preCartan, then there exists U a thin preneighborhood of x.

The next step is to show that $G_x \subseteq ((U, U))$.

Let $g \in G_x$ then $gx = x$ which leads to $gU \cap U \neq \varnothing$.

Then $g \in ((U, U))$. Hence $G_x \subseteq ((U, U))$ which is relatively compact and by 2.4 we get that $G_x$ is closed in G. Then $G_x$ is compact.

***Proposition 3.6:***

If X is a preCartan G-space and $x \in X$, then $g \rightarrow gx$ is a preopen map of G onto Gx.

***Proof:***

Let U be a preopen subset of G.

To prove that Ux is preopen in Gx. (i.e. (G-U)x is preclosed in Gx).

Let $y \in \overline{(G-U)x}^p$. Then by 2.8, there is a net $(g_\alpha x)$ in (G-U)x such that $g_\alpha x \overset{p}{\propto} y$. Since X is preCartan, then there exists V a thin preneighborhood of y.

By 3.1, there is a subnet ($g_{\alpha_{\mu}}$ x) of the net ($g_\alpha$x) in V such that $g_{\alpha_{\mu}}$ x→y.

Fixing $\alpha_o$, then ($g_{\alpha_{\mu}} g_{\alpha_o}^{-1}$)( $g_{\alpha_o}$ x)=$g_{\alpha_{\mu}}$ x.

As in the proof of 2.4(a), then $g_{\alpha_{\mu}} g_{\alpha_o}^{-1}$ ∈ ((V, V)).

Since ((V, V)) is relatively compact, then by 2.1(ii), ($g_{\alpha_{\mu}} g_{\alpha_o}^{-1}$ ) has a cluster point say g.

Hence by 2.1(iii), ($g_{\alpha_{\mu}} g_{\alpha_o}^{-1}$ ) has a subnet which converges to g.

So $g_{\alpha_{\mu}} g_{\alpha_o}^{-1}$ →g, then $g_{\alpha_{\mu}}$ →g$g_{\alpha_o}$ and by 2.2, we get $g_{\alpha_{\mu}}$ x → g $g_{\alpha_o}$ x.

Since U is open and $g_{\alpha_{\mu}}$ ∉ U, then by 2.3, we have g $g_{\alpha_o}$ ∉ U.

So g $g_{\alpha_o}$ ∈ G-U.

Since X is $T_2$, then by 2.1(i), we have y = g$g_{\alpha_o}$ x∈(G-U)x.

Hence $\overline{(G-U)x}^{\,p} \subseteq (G-U)x$.

But we have $(G-U)x \subseteq \overline{(G-U)x}^{\,p}$.

Therefore $(G-U)x = \overline{(G-U)x}^{\,p}$.

Then by 2.7, we get that (G-U)x is preclosed.
Hence Ux is preopen in Gx.

***Theorem 3.7:***

Let X and Y be G-spaces and let λ:X→Y be an onto, strongly preopen and equivariant function. If X is a semi Cartan G-space, then so is Y.

***Proof:***

Let y∈Y. Since λ is onto, then there exists x∈X such that λ(x) = y.

Since X is a preCartan G-space and x∈X, then x has U as a thin preneighborhood.

Since λ is strongly preopen, then λ(U) is a preneighborhood of y. To show that λ(U) is thin we have to prove that ((U, U)) = ((λ(U), λ(U))).

g∈ ((U, U))↔gU∩ U ≠∅↔λ (gU∩ U) ≠∅↔ since λ is onto λ (gU) ∩λ (U) ≠∅↔ since λ is equivariant gλ(U)∩λ (U) ≠∅↔g∈ ((λ(U), λ(U))). Hence:

((U, U))= ((λ(U), λ(U))).

Because ((U, U)) is relatively compact, then so is ((λ(U), λ(U))).Hence Y is a preCartan G-space.

***Proposition 3.8:***

If X is a preCartan G-space, H is a closed subgroup of G and Y is an preopen subspace of X which is an H-invariant subspace of X, then Y is a preCartan H-space.

***Proof:***

By [1] (H,Y) is a topological transformation group. Since Y is a subspace of X and X is a completely regular Hausdorff space, then so is Y. Since G is locally compact and H is a closed subgroup of G, then by [9] H is locally compact.

Hence Y is an H-space.

At the present time we are going to prove that Y is preCartan. Let y∈Y. Then y∈ X.

Since X is a preCartan G-space then y has U as a thin preneighborhood in X. Let U'=U∩Y.

Since Y is a preopen subspace of X, then by 2.6 we get that U' is a preneighborhood of y in Y.

So by [2] U' is a thin preneighborhood of y in Y.

Hence Y is a preCartan H-space.

***Proposition 3.9:***

Let X and Y be G-spaces. Then X × Y is a preCartan G-space if at least one of X or Y is preCartan.

***Proof:***

At first we shall show that X × Y is a G-space.

Since X is a G-space, then G acts on X by $\pi_1$:G×X→X such that $\pi_1$ (g, x) = gx for each g∈G and x∈X. Since Y is a G-space, then G acts on Y by $\pi_2$:G×Y→Y such that $\pi_2$ (g, y) = gy for each g∈G and y∈Y.

Define π: G×X×Y → X×Y such that:

π(g, (x, y)) = g(x, y) = (gx, gy) for each g∈G, x∈X and y∈Y.

a) π is continuous.

b) π (e, (x, y)) = e (x, y) = (ex, ey) = (x, y)

c) $\pi(g_1, \pi(g_2, (x, y)) = \pi (g_1, g_2 (x, y))$
$= g_1g_2(x, y)$
$= (g_1g_2x, g_1g_2y)$
$= \pi(g_1g_2, (x, y))$

Hence X × Y is a G-space.

Now to prove that X × Y is preCartan.

Let (x, y)∈ X × Y.

Since x∈X and X is preCartan, then there exists U a thin preneighborhood of x.

By 2.5 we get U × Y as a preneighborhood of (x, y) in X × Y.

Because we have $((U, U)) = ((U\times Y, U\times Y))$.
So, $((U\times Y, U\times Y))$ is relatively compact, which means that $X \times Y$ is a preCartan G-space.

***Theorem 3.10:***

If a G-space X has a star thin preopen set U, then X is a preCartan G-space.

***Proof:***

Let $x \in X$.
Since U is a star set, then there is $g \in G$ such that $gx \in U$.
Hence $x \in g^{-1} U$.
Since $\pi_g: X \to X$ is strongly preopen for each $g\in G$, then $g^{-1}U$ is a preopen set of x.
Since U is thin, then by [2] we get that $((g^{-1} U, g^{-1} U))$ is relatively compact in G.
That is $g^{-1} U$ is a thin preneighborhood of x in X
Thus X is a preCartan G-space.

***Theorem 3.11:***

If X is a preCartan G-space, then:
(a) There is no fixed point.
(b) There is no periodic point.

***Proof:***

a)Let $x \in X$ such that x is a fixed point.
Since X is a preCartan G-space, then x has U as a thin preneighborhood in X.
Because x is a fixed point, then $gx= x$ for each $g\in G$.
So $gU\cap U \neq \emptyset$ for each $g\in G$.
That is $((U, U)) = G$.
Since $((U, U))$ is relatively compact in G, then G is compact.
But G is not compact, which leads to a contradiction.
Hence X has no fixed point.
(b) Let $x \in X$ such that x is a periodic point.
Then $G_x$ is a syndetic subgroup in G.
That is there is a compact subset K of G such that $G = G_x K$.
By 3.5(b) $G_x$ is compact in G for each $x \in X$.
Thus G is compact
But that leads to a contradiction since G is not compact.
Hence X has no periodic point.

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**الخلاصة**

فضاء– G لبريكارتان هو هدفنا الاساسي بهذا البحث حيث حصلنا على النتائج التالية:

(i) فضاء – G لكارتان هو فضاء – G لبريكارتان.

(ii) قدمنا بعض النتائج على الشبكة مع مجموعة preopen.

(iii) قدمنا هذا الفضاء (فضاء –G لبريكارتان) مع امثلة ونظريات وافية عنه. حيث درسنا خصائصه، فضائه الجزئي، جدائه، وصورة التكافؤ المتغاير له.