On Some Properties of Maximal M-Open Sets

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Abstract

In this work we introduce maximal m-open set in minimal structure spaces and study some of their basic properties in these spaces.

1. Introduction

Recall that a subset U of a topological space X is said to be maximal open set if any open set which contains U is X or U. This concept is introduced by Nakaoka F. and Oda N. in [1]. Then they gave other equivalence definition for the maximal open set by lemma 2.2.1 [a subset U of a topological space X is maximal open set if any open set W, then $U \cup W = X$ or $W \subseteq U$].Also if X is a non empty set, then by minimal structure space (X, m_X) (or m_X-space) means a collection m_X of subsets of X such that X, $\emptyset \in m_X$, members of m_X called m-open and their complements called m-closed and the m-closure (m-interior) of a subset A of X denoted by m-Cl(A) (m-Int(A)) is defined as: m-Cl(A) = $\cap\{U: U \in m_X, A \subseteq U\}$(m-Int(A) = $\cup\{U: U \in m_X, U \subseteq A\}$) [2]. The aim this work is to give the concept of maximal m-open set. Some theorems and properties relative to this concept are introduced.

2. Maximal m-Open Sets

In this section we give the definition of maximal m-open subset of minimal structure space (X, m_X). Throughout this section, we assume (X, m_X) to be minimal structure space.

Definition 2.1:

A proper nonempty m- open subset U of X is said to be a maximal m- open set if for any m-open subset W of X, then $U \cup W = X$ or $W \subseteq U$.

Lemma 2.2:

(i) If a subset U of X is maximal m-open, then any m-open set which contains U is X or U.

(ii) If U and V are maximal m-open sets, then either U = V or $U \cup V = X$.

Proof:

(i) Suppose that there is an m-open set W of X such that $U \subsetneq W \subsetneq X$, then $U \cup W = W$ but U is maximal m-open, then $U \cup W = X$ so W=X which is a contradiction. Therefore any m-open which contains U is X or U.

(ii) It follows directly from the Definition 2.1.■

Remark:

The converse of lemma 2.2.(i), is not true in general. For example if X={1,2,3,4,5} and m_X={X,∅,{1,2,3},{4,5},{4}}, then any m-open set which contains {1,2,3} is X or {1,2,3}. But {1,2,3} is not maximal m-open set.

Definition 2.3:

Let (X, m_X) be a minimal structure. A subset W of X is said to be an m-neighborhood of $x \in X$, if there exists $U \in m_X$ and $x \in U \subseteq W$.

Remark:

In above definition if also $W \in m_X$, then W is called m-open neighborhood.

Proposition 2.4:

Let U be a maximal m-open set. If x is an element of U, then for any m-open neighborhood W of x, $W \cup U = X$ or $W \subseteq U$.

Proof:

It follows from the fact that any m-open neighborhood is m-open.■

Theorem 2.5:

Let U, V, and W be maximal m-open sets such that $U \neq V$. If $U \cap V \subseteq W$, then U = W or V=W.

Proof:

Suppose neither that $U \neq W$ nor $V \neq W$. Since W is maximal m-open set then X–U and X–V are subsets of W and so

$(X-U)\cup(X-V)= X-(U\cap V) \subseteq W$ but $U\cap V$ sets, $w\subseteq W$. Then we have $W=X$ is a contradiction. Therefore $U=W$ or $V=W$. ■

Theorem 2.6:

Let U, V, and W be maximal m-open which are different from each other. Then, $U\cap V\nsubseteq U\cap W$. ■

Proof:

The proof follows directly from Theorem 2.5. ■

Proposition 2.7:

Let U be a maximal m-open set and x an element of U. Then, $U = \cup\{W \mid W$ is an m-open neighborhood of x such that $W\cup U \neq X\}$.

Proof:

It is follows from the fact that U is an m-open neighborhood of x, we have $U \subseteq \cup\{W|W$ is an m-open neighborhood of x such that $W \cup U \neq X\}$.Since U is maximal m-open then $\cup\{W \mid W$ is an m-open neighborhood of x such that $W\cup U \neq X\} \subseteq U$. Therefore, we have the result.■

3. m-Closure, m-Interior, and Maximal m-Open Sets

In this section we computation the closure of maximal m-open sets and the m-closure, the m-interior of other sets.

Theorem 3.1:

Let U be a maximal m-open set and x be an element of $X-U$. Then, $X-U\subseteq W$ for any m-open neighborhood W of x.

Proof:

Suppose that $X-U\nsubseteq W$, for some m-open neighborhood W of x. Then $W\cup U\neq X$ which contradicts that U is maximal m-open. Therefore $X-U\subseteq W$.■

Corollary 3.2:

Let U be a maximal m-open set. Then, following (i) or (ii) of the following holds:

(i) For each $x\in X-U$ and each m-open neighborhood W of x, $W=X$;

(ii) There exists an m-open set W such that $X-U\subseteq W$ and $W\subsetneq X$.

Proof:

If (i) does not hold, then there exists an element x of $X-U$ and an m- neighborhood W of x such that $W\subsetneq X$. By Theorem 3.1, we have $X-U\subseteq W$. ■

Corollary 3.3:

Let U be a maximal m-open set. Then, following (i) or (ii) of the following holds:

(i) For each $x\in X-U$ and each m-open neighborhood W of x, we have $X-U\subsetneq W$.

(ii) There exists an m-open set W such that $X-U=W \neq X$.

Proof:

Assume that (ii) does not hold. Assume that $x\in X-U$, then, by Theorem 3.1, we have $X-U\subsetneq W$ each m-open neighborhood W of x.■

Theorem 3.4:

Let U be a maximal m-open set. Then, $m\text{-}Cl(U) = X$ or $m\text{-}Cl(U) = U$.

Proof:

Since U is a maximal m-open set, then either one of the following cases (i) and (ii) occur by Corollary 3.3: (i) for each $x\in X-U$ and each m-open neighborhood W of x, we have $X-U\subsetneq W$. In this case let x be any element of $X-U$ and W any m-open neighborhood of x. Since $X-U\neq W$, we have $W\cap U\neq\emptyset$ for any m-open neighborhood W of x. Hence, $X-U\subseteq m\text{-}Cl(U)$.Since $X=U\cup(X-U) \subseteq U\cup m\text{-}Cl(U) = m\text{-}Cl(U) \subseteq X$, we have $m\text{-}Cl(U)=X$; (ii) If there exists an m-open set W such that $X-U=W\neq X$. $X-U =W$ is an m-open set, then U is an m-closed set. Therefore, $U=m\text{-}Cl(U)$.■

Theorem 3.5:

Let U be a maximal m-open set. Then $m\text{-}Int(X -U) = X -U$ or $m\text{-}Int(X-U) = \emptyset$.

Proof:

By Corollary 3.3, we have either (i) $m\text{-}Int(X -U) = \emptyset$ or (ii) $m\text{-} Int(X-U) = X-U$.■

Theorem 3.6:

Let U be a maximal m-open set and S a nonempty subset of $X-U$. Then $m\text{-}Cl(S) = X-U$.

Proof:

Since $\emptyset \neq S \subset X-U$, we have $W \cap S \neq \emptyset$ for any element x of $X-U$ and any m-open neighborhood W of x by Theorem 3.1. Then $X-U \in$ m-Cl(S). Since $X-U$ is an m-closed set and $S \subset X-U$, then m-Cl(S) $\subset$ m-Cl($X-U$) = $X-U$.■

Corollary 3.7:

Let U be a maximal m-open set and M a subset of X with $U \subsetneq M$. Then, m-Cl(M)=X.

Proof:

Since $U \subsetneq M \subset X$, there exists a nonempty subset S of $X-U$ such that $M = U \cup S$. Hence, we have m-Cl(M) = m-Cl($S \cup U$)=m-Cl(S)$\cup$m-Cl(U) $\supset$ $(X-U) \cup U$ = X by Theorem 3.6. Therefore, m-Cl(M)=X.■

Corollary 3.8:

Let U be a maximal m-open set and assume that the subset $X-U$ has at least two elements. Then, m-Cl($X-\{a\}$) = X for any element a of $X-U$.

Proof:

Since $U \subsetneq X-\{a\}$ by our assumption, we have the result by Corollary 3.7.■

Theorem 3.9:

Let U be a maximal m-open set and N a proper subset of X with $U \subseteq N$. Then m-Int(N) = U.

Proof:

If N = U, then m-Int(N) = m- Int(U) = U. Otherwise $N \neq U$, and hence $U \subsetneq N$. It follows that $U \subset$ m-Int(N). Since U is a maximal m-open set, we also m-Int(N) $\subset$ U. Therefore, m-Int(N) = U.■

Theorem 3.10:

Let U be a maximal m-open set and S a nonempty subset of $X-U$. Then $X-$m-Cl(S) = m-Int($X-S$) = U.

Proof:

Since $U \subseteq X-S \subsetneq X$ by our assumption, we have the result by Theorems 3.6 and Theorem 3.9. ■

Definition 3.11:

A subset M of a space (X, m_X) is called an m-preopen set if $M \subset$ m-Int (m-Cl(M)).

Theorem 3.12:

Let U be a maximal m-open set and M any subset of X with $U \subset M$. Then, M is a m-preopen set.

Proof:

If M = U, then M is an m-open set. Therefore, M is an m-preopen set. Otherwise, $U \subsetneq M$, then m-Int(m-Cl(M)) = m-Int(X) = $X \supset M$ by **Corollary 3.7**. Therefore, M is an m-preopen set.■

Corollary 3.13:

Let U be a maximal m-open set. Then, $X-\{a\}$ is an m-preopen set for any element a of $X-U$.

Proof:

Since $U \subset X-\{a\}$ by our assumption, we have the result by Theorem 3.12.■

4. Fundamental Properties of Radicals

In this section, we introduce the concept of radicals of maximal m-open sets and some of its properties.

Definition 4.1:

Let U_λ be a maximal m-open set for any element λ of Λ. Let $\mu = \{U_\lambda \mid \lambda \in \Lambda\}$, $\cap\mu = \cap\{U_\lambda : \lambda \in \Lambda\}$ is called the radical of μ. The intersection of all maximal ideals of a ring is called the (Jacobson) radical of μ [3]. Following this terminology in the theory of rings, we use the terminology "radical" for the intersection of maximal m-open sets. The symbol $\Lambda \backslash \Gamma$ means difference of index sets; namely, $\Lambda \backslash \Gamma = \Lambda - \Gamma$, and the cardinality of a set Λ is denoted by $|\Lambda|$ in the following arguments.

Theorem 4.2:

Assume that $|\Lambda| \geq 2$. Let U_λ be a maximal m-open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. (i) Let μ be any element of Λ. Then, $X - \cap_{\lambda \in \Lambda \backslash \{\mu\}} U_\lambda \subset U_\mu$. (ii) Let μ be any element of Λ. Then, $\cap_{\lambda \in \Lambda \backslash \{\mu\}} U_\lambda \neq \emptyset$.

Proof:

Let μ be any element of Λ. (i) Since $X-U_\mu \subset U_\lambda$ for any element λ of Λ with $\lambda \neq \mu$. Then, $X - U_\mu \subset \cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda$ Therefore, we have $X-\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda \subset U_\mu$. (ii) If $\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda = \emptyset$,we have $X = U_\mu$ by (i). This contradicts our assumption that U_μ is maximal m-open set. Therefore, we have $\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda \neq \emptyset$.■

Corollary 4.3:

Let U_λ be a maximal m-open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $|\Lambda| \geq 3$, then $U_\lambda \cap U_\mu \neq \emptyset$ for any elements λ and μ of Λ with $\lambda \neq \mu$.

Proof:

By Theorem 4.2(ii), we have the result.■

Corollary 4.4:

(a decomposition theorem for maximal m-open set). Assume that $|\Lambda| \geq 2$. Let U_λ be a maximal m-open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. Then, for any element μ of Λ, $U_\mu = (\cap_{\lambda\in\Lambda}U_\lambda)\cap(X-\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda)$.

Proof:

Let μ be an element of Λ. By Theorem 4.2 (1), we have:

$(\cap_{\lambda\in\Lambda}U_\lambda)\cup(X-\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda)=((\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda)\cap U_\mu)\cup(X-\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda)=((\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda)\cup(X-\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda))\cap(U_\mu\cup(X-\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda))=U_\mu\cup(X-\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda)=U_\mu$. Therefore, we have $U_\mu=(\cap_{\lambda\in\Lambda}U_\lambda)\cup(X-\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda)$.■

Theorem 4.5:

Let U_λ be a maximal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. Assume that $|\Lambda| \geq 2$. Let μ be any element of Λ. Then, $\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda \not\subset U_\mu \not\subset \cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda$.

Proof:

Let μ be any element of Λ. If $\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda \subset U_\mu$, then we see that $X=(X-\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda)\cup\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda \subset U_\mu$ by Theorem 4.2(i). This contradicts our assumption. If $U_\mu\subset\cap_{\lambda\in\Lambda\backslash\{\mu\}}U_\lambda$, then we have $U_\mu \subset U_\lambda$, and hence $U_\mu=U_\lambda$ for any element λ of $\Lambda\backslash\{\mu\}$. This contradicts our assumption that $U_\mu\neq U_\lambda$ when $\lambda\neq\mu$.■

Corollary 4.6:

Let U_λ be a maximal m-open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If Γ is a proper nonempty subset of Λ, then $\cap_{\lambda\in\Lambda\backslash\Gamma}U_\lambda \not\subset \cap_{\gamma\in\Gamma}U_\gamma \not\subset \cap_{\lambda\in\Lambda\backslash\Gamma}U_\lambda$.

Proof:

Let γ be any element of Γ. We see $\cap_{\lambda\in\Lambda\backslash\Gamma}U_\lambda=\cap_{\lambda\in((\Lambda\backslash\Gamma)\cup\{\gamma\})\backslash\{\gamma\}}U_\lambda \not\subset U_\gamma$ by Theorem 4.5. Therefore we see $\cap_{\lambda\in\Lambda\backslash\Gamma}U_\lambda \not\subset \cap_{\gamma\in\Gamma}U_\gamma$. On the other hand, since $\cap_{\gamma\in\Gamma}U_\gamma = \cap_{\gamma\in\Lambda\backslash(\Lambda\backslash\Gamma)}U_\gamma \not\subset \cap_{\lambda\in\Lambda\backslash\Gamma}U_\lambda$, we have $\cap_{\gamma\in\Gamma}U_\gamma \not\subset \cap_{\lambda\in\Lambda\backslash\Gamma}U_\lambda$.■

Theorem 4.7:

Let U_λ be a maximal m-open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If Γ is a proper nonempty subset of Λ, then $\cap_{\lambda\in\Lambda}U_\lambda \subsetneq \cap_{\gamma\in\Gamma}U_\gamma$.

Proof:

By Corollary 4.6, we have $\cap_{\lambda\in\Lambda}U_\lambda = (\cap_{\lambda\in\Lambda\backslash\Gamma}U_\lambda) \cap (\cap_{\gamma\in\Gamma}U_\gamma) \subseteq \cap_{\gamma\in\Gamma}U_\gamma$.■

Theorem 4.8:

Assume that $|\Lambda| \geq 2$. Let U_λ be a maximal m-open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\cap_{\lambda\in\Lambda}U_\lambda = \emptyset$, then $\{U_\lambda \mid \lambda \in \Lambda\}$ is the set of all maximal m-open sets of X.

Proof:

If there exists another maximal m-open set U_ν of X, which is not equal to U_λ for any element λ of Λ, then $\emptyset = \cap_{\lambda\in\Lambda}U_\lambda = \cap_{\lambda\in(\Lambda\in\{\nu\})\backslash\{\nu\}}U_\lambda$. By Theorem 4.2(ii), we see $\cap_{\lambda\in(\Lambda\in\{\nu\})\backslash\{\nu\}}U_\lambda \neq \emptyset$. This contradicts our assumption.■

Proposition 4.9:

Let U_λ be a set for any element λ of Λ. If $m\text{-}Cl(\cap_{\lambda\in\Lambda}U_\lambda) = X$, then $m\text{-}Cl(U_\lambda) = X$ for any element λ of Λ.

Proof:

We see that $X = m\text{-}Cl(\cap_{\lambda\in\Lambda}U_\lambda) \subset m\text{-}Cl(U_\lambda)$. It follows that $m\text{-}Cl(U_\lambda) = X$ for any element λ of Λ.■

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الخلاصة

في هذا العمل نحن نقدم مفهوم المجموعة m− المفتوحة العظمى في فضاءات البنية الصغرى و ندرس بعض الخصائص الأساسية لهذه المجموعة في تلك الفضاءات.