On Representation of Monomial Groups

S.A.Bedaiwi and A.A.Hajim
Department of Mathematics, College of Science, Al-Mustansiryah University, Baghdad-Iraq.

Abstract

Taketa shows that all monomial groups (commonly written as M-groups) are solvable. Gajendragadkar gives the notion of π-factorable character. We show that an irreducible character of an M-group is primitive if it is π-factorable. Issacs proves that product of two monomial characters is a monomial. We extend this fact to include any finite number of monomial characters consequently we prove that any product of finite number of M-groups is an M-group. We show that any group of order 45 is an M-group and for any group G, the factor group G/G' is an M-group.

Keywords: Representation theory, Monomial groups, π-factorable characters.

1. Introduction

The essential body of representation theory has been constructed by Richard Brauer (1901-1977). His processors; Frobenius, Burnside and Schur, gave the grand task to which character theory could make a central contribution, that is, the complete classification of finite simple groups [1], [2], [13].

T.Okuyanta [12] proved that if G is an M-group and P is Sylow P-subgroup of G, then $N_G(P)/P$ is an M-group. I.M.Issacs [7] shows that if H is a Hall subgroup of an M-group then $N_G(H)/H'$ is also and M-group.

In studying monomial groups it is important to know as much as possible about the primitive characters of its subgroups, since that every character is induced from a primitive character [4].

The following are proved:

- Any irreducible character of monomial group is primitive if it is π-factorable.
- Any finite product of monomial characters is monomial.
- The external direct product of n-copies of monomial groups is monomial.

2. Characters and M-groups

Character theory was developed by Frobenius in 1896. It provides a powerful tool for proving theorems about finite groups. No non-character theoretic description of the class of M-groups has been found. We use character techniques to gain more information and facts about M-groups.

2.1 Definition [8]:

Let χ be a character of G, then χ is monomial if $c = l^G$ where λ is a linear character of some subgroup of G.

2.2 Definition [7]:

Let G be any group, we denote by $Irr(G)$ for the set of all irreducible characters of G.

2.3 Definition [8]:

A group G is an M-group (monomial group) if every $c \in Irr(G)$ is monomial character.

2.4 Theorem (Taketa) [1]:

Every M-group is solvable

2.5 Theorem [8]:

Every nilpotent group is an M-group.

2.6 Definition [2]:

Let $p = \{p_1, p_2, ..., p_n\}$ be a non-empty set of primes a π-number is a positive integer whose prime divisors belong to π. An element of a group is called a π-element if its order is a π-number and if every element of a group is π-element, the group is called π-group.

2.7 Remark:

Let π be a set of primes define π' to be the complement primes of π, the π'-number is a positive integer whose prime divisors does not belong to π. An element of a group is called a π'–element if its order is a π'-number and if every element of a group is π'-element the group is called π'-group.

2.8 Definition [9]:

Let G be a finite group and let π be a nonempty set of primes. Then G is said to be π-separable if it has normal series each factor of which is either a π-group or π'-group.

2.9 Definition [7]:

Let χ be a character of G and det χ = l be the uniquely defined linear character, write $o(c) = o(l)$ the order of l as an element of the group of linear characters of G is called the determinantal order of χ

2.10 Definition [6]:

Let $c \in Irr(G)$, and let π be a set of primes. Then χ is π-special, (π'-special) provided that χ (1) is a π-number (π'-number) and that for all subnormal subgroups $S << G$ and all irreducible constituents ɵ of c_S, the determinantal order O(ɵ) is a π –number, (π'-number).

2.11 Definition [6]:

Let $c \in Irr(G)$, we say that χ is π-factorable if there exist $z, h \in Irr(G)$, z is p-special and h is p'-special such that $c = zh$.

2.12 Definition [8]:

Irreducible characters whose restriction to every normal subgroup is homogeneous (multiples of an irreducible) are called quasi-primitive.

2.13 Theorem [6]:

Let G be a π-separable. Then every quasi-primitive $c \in Irr(G)$ is π-factorable.

2.14 Definition [8]:

Let G be any group, $N < G$, $q \in Irr(N)$ then q is called primitive if it cannot be obtained by inducing any character of proper subgroup.

2.15 Proposition [6]:

Let G be π-separable and let $c \in Irr(G)$ be primitive. Then χ factors as a product of primitive π-special and π'-special characters.

2.16 Lemma[9]:

Let G be any group, $x, h \in Irr(G)$ are monomial and $c = xh \in Irr(G)$ then χ is monomial.

3 Main Results and Applications

3.1 Definition [7]:

If $c = \sum_{i=1}^{k} n_i c_i$, then those c_i with n_i **f** 0 are called the irreducible constituent of c

3.1 Proposition:

Let G be a π-separable group, $c \in Irr(G)$ is quasi-primitive. Then the π-special and π'-special factors of χ are quasi-primitive.

Proof:

We can write $c = zh$ where z is π-special and h is π'-special. Let $N < G$ and let a *and* b be irreducible constituent of z_N *and* h_N respectively, then ab is irreducible and is a constituent of c_N .Since χ is quasi-primitive it follows that ab is G-invariant and thus a *and* b are G-invariant by the uniqueness of factorization.

3.2 Theorem:

Let G be an M-group. Then $c \in Irr(G)$ is primitive if it is π-factorable.

Proof:

Let $c \in Irr(G)$ be a primitive, since G is M-group then by theorem 2.4 G is solvable and hence π-separable. Since χ is primitive it is quasi-primitive and by theorem 2.13 χ is π-factorable.

3.3 Remark:

Any finite product of monomial characters is monomial.

3.4 Proposition:

External direct product of n-copies of monomial group is monomial.

Proof:

Let G_i be monomial group for each i, to show that $\prod_{i=1}^{n} G_i$ is monomial. Let $c = \prod_{i=1}^{n} h_i \in Irr(\prod_{i=1}^{n} G_i)$ where $h_i \in G_i$, since G_i is monomial group for each i, by Definition 2.1 h_i is monomial character for each i, by Remark 3.3 $c=\prod_{i=1}^{n} h_i$ is monomial therefore $\prod_{i=1}^{n} G_i$ is monomial group.

3.5 Proposition: Any group of order 45 is an M-group.

***Proof*:**
Let G be any group of order 45, since $45 = 3^2.5$ G has a 3-sylow subgroup H of order 9 and a 5-sylow subgroup K of order 5. Let n is the number of the distinct conjugates of H, then n=1+3r $(r \geq 0)$ and n divides 45 the only possibility is r=0 thus n=1 and hence H is normal in G. Similarly K is normal in G, we have G=HK. Since $|HK| = |HK||H \cap K| = |H||K| = 45$ thus G isomorphic to $H \times K$ but H is abelian and K is cyclic [11] so G is abelian and hence it is nilpotent therefore by theorem 2.5 it is an M-group.

3.6 Proposition: Let G be a group and let G' be the derived subgroup of G, then G/G' is an M-group.

***Proof*:**
We know that G' is normal in G, let $x, y \in G$ then;

$(xG')(yG') = xyG' = xy[(x^{-1}y^{-1}yx)G'] = (xyx^{-1}y^{-1})yxG' = yxG' = yG'xG'$

since $(xyx^{-1}y^{-1}) \in G'$ thus G/G' is abelian and hence nilpotent (in fact every abelian group is nilpotent group of class one) therefore it is M-group by theorem 2.5.

<u>3.7 Proposition:</u>
The quotient group $GL(2,R)/SL(2,R)$ is an M-group.

***Proof*:**
We show that $SL(2,R)$ is the derived subgroup of $GL(2,R)$ and by using proposition 3.9 we are done. The mapping $GL(2,R) \rightarrow$ defined by $x \rightarrow \det(x)$ is a homomorphism with kernel $SL(2,R)$, thus the special linear group is a normal subgroup and $[GL(2,R)]' \subseteq SL(2,R)$.

Now, the following matrices are the generators of $SL(2,R)$:

$$\begin{pmatrix}1 & r\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\s & 1\end{pmatrix}, \begin{pmatrix}t & 0\\0 & 1/t\end{pmatrix}, \qquad (r,s,t \in \quad , t \neq 0)$$

Where

if $\begin{pmatrix}a & b\\c & d\end{pmatrix} \in SL(2,R)$ we have two cases; when

$c \neq 0$

$$\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}1 & (a-1)/c\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\c & 1\end{pmatrix}\begin{pmatrix}1 & (d-1)/c\\0 & 1\end{pmatrix}$$

When $c = 0$

$$\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}a & 0\\0 & 1/a\end{pmatrix}\begin{pmatrix}1 & b/a\\0 & 1\end{pmatrix}$$

And the calculation below show that the generators of SL (2,R) are commutators

$$\begin{pmatrix}1 & -r\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}1 & r\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}1 & r\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\s & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}1 & 0\\-s & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\s & 1\end{pmatrix}$$

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}t & 2t\\2t^2 & t^2\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}-1/3t & 2/3t^2\\2/3t & -1/3t^2\end{pmatrix} = \begin{pmatrix}t & 0\\s & 1/t\end{pmatrix}$$

And therefore we have:
$SL(2,R) \subseteq [GL(2,R)]'$

Acknowledgment
We would like to express our deep gratitude for the referee (s) for the suggested valuable comments.

References
[1] C.W.Cutis and I. Reiner; Representation theory of finite groups and associative algebras. Wiley-Interscience, NewYork, 1962.
[2] C.W.Curtis; Methods of representations theory with applications to finite groups and order. John and sons, NewYork, 1981.
[3] D.Gajendragadkar; A characteristic class of characters of finite π-separable groups, Journal of algebra 95, 237-259, 1979.
[4] G. Navaro; Primitive characters of subgroups of M-groups, Math. Z., 218, 439-445, 1995.
[5] G. Qian; Finite groups with consecutive nonlinear character degrees, Journal of Algebra, Vol, 285, 372-382, 2005.
[6] I.M. Issacs; Primitive characters, Normal subgroups, and M-groups, Math.Z., 177, 268-284, 1981.

[7] I.M. Isaacs; Hall subgroups normalizers and character correspondences in M-groups, Proceedings of the A.M.S, Vol.109, 647-651, 1990.
[8] I.M. Isaacs; Character theory of finite groups. Academic press, INC. 1976.
[9] I.M.Isaacs; Characters of subnormal subgroups of M-groups, ARCH. Math. Vol.42, 509-515, 1984.
[10] M.L. Lewis; Primitive characters of subgroups of M-groups, Proceedings of the A.M.S, Vol, 125, 27-33, 1997.
[11] J. Rotman; An introduction to the theory of groups, Springer-Verlag, 1995.
[12] Okuyanta. T; Model correspondence in finite groups, Hokkaido Math J., 10, 299-318, 1981.
[13] W.Fulton and J.Harris; Representations theory, GTM 129, Springer-Verlag, 1991.