Spectral Theory in Fuzzy Normed Spaces

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Abstract

In this paper we give some definitions related to spectral theory of a linear operator T defined on a fuzzy normed space and we prove that spectrum σ(T) and resolvent set ρ(T) are nonempty for a fuzzy bounded linear operator defined on certain fuzzy normed spaces. Moreover, we show ρ(T) is τ-open and σ(T) is τ-closed.

Keywords: fuzzy norm, fuzzy bounded linear operator, spectrum of a fuzzy bounded linear operator, resolvent set of a fuzzy bounded linear operator

1.Introduction

In 1965, Zadeh [6] defined a fuzzy set as a class of objects with a continuum of grades of membership. Such a set in characterized by a membership function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion union, intersection, complement, relation, etc. are extended to such sets. This concept has been studied intensively by many authors [1], [2], [3], [4], [7], etc.. The purpose of this paper is to generalized the concept of spectral theory [5] to the concept of fuzzy theory. To do this, we need to recall the following definitions. Let X be a linear space over a field K. A fuzzy subset N of $X\times\Re$ is said to be a fuzzy norm on a linear space X in case for each x, $y\in X$ and $c\in K$, the following conditions hold

(N_1) N(x,t)=0 for each $t\leq 0$.

(N_2) N(x,t)=1 for each t>0 if and only if x= **0.**

(N_3)If $0\neq c\in K$ then $N(cx,t)=N\left(x,\frac{t}{|c|}\right)$ for each t>0

(N_4) $N(x+y,s+t)\geq N(x,s)\wedge N(y,t)$ for each s, $t\in\Re$.

(N_5) $N(x,\cdot)$ is a non-decreasing function of $\Re$ and $\lim\limits_{t\to\infty} N(x,t)=1$.

The pair (X,N) will be referred to as a fuzzy normed space [9]. Also, recall that a subsetU of a fuzzy normed space (X,N) is said to be he closure of a subset V of X in case for any $x\in U$, there exists a sequence $\{x_n\}$ in V such that $\lim\limits_{n\to\infty} N(x_n - x,t)=1$ for each t>0. We denote the set U by $\overline{V}$, [9]. On the other hand a subset U of a fuzzy normed space (X,N) is said to be dense in case $\overline{U}=X$, [9]. Also, recall that a linear operator $T:(X,N)\longrightarrow(X,N)$ is said to be fuzzy bounded on X in case there exists D>0 such that for each $x\in X$ and $t\in\Re$, N(T(x),t)) $\geq N\left(x,\frac{t}{D}\right)$, [8].

2. The Main Results

We start this section by giving the following two propositions give the relation between the ordinary closure and the closure in the fuzzy sense in case the fuzzy norm N.

Proposition (2.1):

Let $(X,\|\cdot\|)$ be a normed space and let N be the fuzzy norm defined by

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|} & \text{for} \quad t>0 \\ 0 & \text{for} \quad t\leq 0 \end{cases} \qquad \ldots\ldots\ldots (2.1)$$

for each $x\in X$. Then the closure of a subset U of X with respect to $\|.\|$ is equal to the closure of U with respect to N.

Proof :

Suppose V is the closure of U with respect to $\|.\|$. Then for each $x \in V$, there exists a sequence $\{x_n\}$ in U such that $\lim_{n\to\infty} \|x_n - x\| = 0$. Hence, for each t>0, $\lim_{n\to\infty} N(x_n - x, t) = 1$. Thus each element of V belongs to the closure of U with respect to N.

Conversely, suppose $\overline{U}$ is the closure of U with respect to N. Then for each $x \in \overline{U}$ these exists a sequence $\{x_n\}$ in U such that for each t>0, $\lim_{n\to\infty} N(x_n - x, t) = 1$. Hence $\lim_{n\to\infty} \|x_n - x\| = 0$. Thus each element of $\overline{U}$ belongs to V. Therefore $V = \overline{U}$.

Proposition (2.2):

Let $(X, \|.\|)$ be a normed space and let N be the fuzzy norm defined by

$$N(x,t) = \begin{cases} 0 & \text{for } t \le \|x\| \\ 1 & \text{for } t > \|x\| \end{cases} \quad \text{.....................} \ (2.2)$$

for each $x \in X$ and $t \in \Re$. Then the closure of a subset U of X with respect to $\|.\|$ is equal to the closure of U with respect to N.

Proof :

Suppose V be the closure of U with respect to $\|.\|$ then for each $x \in V$ there exists a sequence $\{x_n\}$ in U such that $\lim_{n\to\infty} \|x_n - x\| = 0$. That is for each $\varepsilon > 0$, there exists a positive integer n_0 such that $\|x_n - x\| < \varepsilon$ for each $n \ge n_0$. Therefore $N(x_n - x, \varepsilon) = 1$ for each $n \ge n_0$. Thus $\lim_{n\to\infty} N(x_n - x, \varepsilon) = 1$ for each $\varepsilon > 0$. Therefore, each element of V belong to the closure of U with respect to N.

Conversely, suppose $\overline{U}$ is the closure of U with respect to N. Then for each $x \in \overline{U}$ there exists a sequence $\{x_n\}$ in U such that for each t>0,

$\lim_{n\to\infty} N(x_n - x, t) = 1$. Fix $\alpha \in (0,1)$, thus $\lim_{n\to\infty} N(x_n - x, t) = 1 > \alpha$, for each t>0. That is for each t>0, there exists n_0, such that $N(x_n - x, t) > \alpha$ for each $n \ge n_0$. So $\|x_n - x\| < t$ for each $n \ge n_0$. Hence $\lim_{n\to\infty} \|x_n - x\| = 0$. Thus, each element of $\overline{U}$ belong to V. Therefore $\overline{U} = V$.

Definition (2.3):

Let (X,N) be a fuzzy normed space over the field $\mathbb{C}$ where $X \ne \{\mathbf{0}\}$ and $T: X \longrightarrow X$ be a linear operator. A regular value λ of T is a complex number such that

(1) $R_\lambda(T)$ exists.

(2) $R_\lambda(T)$ is fuzzy bounded linear operator on $R(T_\lambda)$.

(3) $R_\lambda(T)$ is defined on a set which is dense in X.

where $R_\lambda(T) = T_\lambda^{-1} = (T - \lambda I)^{-1}$ call it the resolvent operator of T and $R(T_\lambda)$ the range of T_λ.

Definition (2.4):

Let (X,N) be a fuzzy normed space over the field $\mathbb{C}$ where $X \ne \{\mathbf{0}\}$ and $T: X \longrightarrow X$ be a linear operator. The resolvent set of T, denoted by $\rho(T)$ is the set of all regular values λ of T.

Example (2.5):

Let (X,N) be a fuzzy normed space over the field $\mathbb{C}$ where $X \ne \{\mathbf{0}\}$. It is easy to check that $\rho(I) = \mathbb{C} - \{1\}$ and $\rho(O) = \mathbb{C} - \{0\}$, where I is the identity operator and O is the.

The following propositions show that the resolvent operator $R_\lambda(T)$ of a fuzzy bounded linear operator T defined on certain fuzzy normed spaces is fuzzy bounded on it.

Proposition (2.6):

Let $(X, \|.\|)$ be a complete normed space over the field $\mathbb{C}$ where $X \ne \{\mathbf{0}\}$, $T: X \longrightarrow X$ be a linear operator, N be the fuzzy norm defined by eq.(2.1) and let $\lambda \in \rho(T)$ with respect to (X,N). If T is fuzzy bounded on X then $R_\lambda(T)$ is fuzzy bounded on (X,N).

Proof :

Suppose T is fuzzy bounded on X. Then there exists D>0 such that for each $x \in X$ and

$t\in\Re$, $N(T(x),t)\geq N\left(x,\frac{t}{D}\right)$. Hence for each $x\in X$ and $t>0$, $\frac{t}{t+\|T(x)\|}\geq\frac{t}{t+\|Dx\|}$ implies for each $x\in X$, $\|T(x)\|\leq D\|x\|$. Therefore T is bounded linear operator. Moreover, since $\lambda\in\rho(T)$ with respect to (X,N) then $R_\lambda(T)$ exists, $R_\lambda(T)$ is bounded on $\mathcal{R}(T_\lambda)$ and by using proposition (2.1) one can prove $\mathcal{R}(T_\lambda)$ is dense in $(X,\|.\|)$, thus λ belongs to resolvent set of bounded linear operator T. By [5] one can get $R_\lambda(T)$ is bounded linear operator on $(X,\|.\|)$. then there exists D>0 such that for each $x\in X$, $\|R_\lambda(T)(x)\|\leq D\|x\|$. Then for each t>0 and for each $x\in X$, $t+\|R_\lambda(T)(x)\|\leq t+D\|x\|$. Then there exists D>0 such that for each $x\in X$ and $t\in\Re$, $N(R_\lambda(T)(x),t)\geq N\left(x,\frac{t}{D}\right)$. Therefore, $R_\lambda(T)$ is fuzzy bounded on (X,N).

Proposition (2.7):

Let $(X,\|.\|)$ be a complete normed space over the field $\mathbb{C}$, where $X\neq\{\mathbf{0}\}$, $T:X\longrightarrow X$ be a linear operator, N is the fuzzy norm defined by eq.(2.2) and let $\lambda\in\rho(T)$ with respect to (X,N), if T is fuzzy bounded on X then $R_\lambda(T)$ fuzzy bounded on X.

Proof :

Suppose T is fuzzy bounded on X. Then there exists D*>0 such that for each $x\in X$ and for each $t\in\Re$,

$N(T(x),t)\geq N\left(x,\frac{t}{D^*}\right)$. Assume for the contrary there exists $x^*\neq 0$ such that $\|T(x^*)\|>D^*\|x^*\|$. Let $\|T(x^*)\|=t_0$. Hence $N(T(x^*),t_0)=0$ but $N(D^*x^*,t_0)=1$ this is a contradiction. Then for each $x\in X$ there exists D*>0 such that $\|T(x)\|\leq D^*\|x\|$. Hence T is bounded linear operator. Moreover, since $\lambda\in\rho(T)$ with respect to (X,N) then $R_\lambda(T)$ exists, $R_\lambda(T)$ is bounded on $\mathcal{R}(T_\lambda)$ and by using proposition (2.2) one can prove $\mathcal{R}(T_\lambda)$ is dense in $(X,\|.\|)$, thus λ belongs to resolvent set of bounded linear operator T. By [5] one can get $R_\lambda(T)$ is bounded linear operator on $(X,\|.\|)$. then there exists D>0 such that that for each $x\in X$, $\|R_\lambda(T)(x)\|\leq D\|x\|$. Let $x\in X$ and $t\in\Re$ then we have two cases:-

(1) If $t>\|R_\lambda(T)(x)\|$ then $N(R_\lambda(T)(x),t)=1$. Since $\|R_\lambda(T)(x)\|\leq\|Dx\|$ Then either $t\leq\|Dx\|$ or $\|Dx\|<t$. If $t\leq\|Dx\|$ then $N(Dx,t)=0$. Hence

$N(R_\lambda(T)(x),t)=1>N(Dx,t)=N\left(x,\frac{t}{D}\right)=0$. If $\|Dx\|<t$ then $N(Dx,t)=1$. Thus for each $x\in X$, $N(R_\lambda(T)(x),t)=N(Dx,t)$

$$=N\left(x,\frac{t}{D}\right)=1.$$

(2) If $\|R_\lambda(T)(x)\|\geq t$ then $N(R_\lambda(T)(x),t)=0$. Since $\|R_\lambda(T)(x)\|\leq D\|x\|$ then $N(Dx,t)=0$. Hence

$N(R_\lambda(T)(x),t)=N\left(x,\frac{t}{D}\right)=0$.

Therefore $R_\lambda(T)$ is fuzzy bounded on(X,N).

Proposition (2.8):

Let (X,N) be a fuzzy normed space over the field $\mathbb{C}$ where $X\neq\{0\}$ satisfying the following conditions

(1) For each α, $0<\alpha<1$ and for each sequence$\{x_n\}$ in X satisfying the condition $\lim_{n\to\infty} N(x_{n+p}-x_n,t)\geq\alpha$ for each t>0, p=1,2,…, implies there exists $x\in X$ such that for each t>0, $\lim_{n\to\infty} N(x_n-x,t)>\alpha$.

(2) For each t>0, N(x,t)>0 implies $x=\mathbf{0}$.

(3) For $x\neq\mathbf{0}$, $N(x,\cdot)$ is a continuous function of $\Re$ and strictly increasing on the subset $\{t:0<N(x,t)<1\}$ of $\Re$.

Let $T:X\longrightarrow X$ be a linear operator and let $\lambda\in\rho(T)$. If T is fuzzy bounded on X then $R_\lambda(T)$ fuzzy bounded on X.

Proof :

Since (X,N) satisfied condition (1) one can get X is a complete normed space with respect to $\|.\|_\alpha$ for each $\alpha\in(0,1)$. Since T is fuzzy bounded on X then:

$T:(X,\|.\|_\alpha) \longrightarrow (X,\|.\|_\alpha)$ is bounded for each $\alpha\in(0,1)$. Moreover, since $\lambda\in\rho(T)$ then $R_\lambda(T)$ exists, $R_\lambda(T)$ is bounded on $R(T_\lambda)$ and $R(T_\lambda)$ is dense with respect to $\|.\|_\alpha$ for each $\alpha\in(0,1)$. Thus λ belongs to resolvent set of T with respect to $\|.\|_\alpha$ for each $\alpha\in(0,1)$. Hence $R_\lambda(T):(X,\|.\|_\alpha) \longrightarrow (X,\|.\|_\alpha)$ is bounded linear operator for each $\alpha\in(0,1)$ follows from [5]. Therefore $R_\lambda(T)$ is fuzzy bounded on (X,N).

Remark (2.9) [9]:

Let (X,N) be a fuzzy normed space. Then the set

$\tau=\{G\subset X \mid x\in G$ iff there exist $t>0$ and $\alpha\in(0,1)$ such that $B(x,\alpha,t)\subset G\}$

where $B(x,\alpha,t)=\{y \mid N(x-y,t)>1-\alpha\}$ and for $0<\alpha<1$, is a topology on X.

The following propositions show that $\rho(T)$ is nonempty set and τ-open where τ is the topology induced by the fuzzy norm defined by eq.(2.2).

Proposition (2.10):

Let $(X,\|.\|)$ be a complete normed space over the field $\mathbb{C}$, where $X\neq\{\mathbf{0}\}$ and let $T:(X,N) \longrightarrow (X,N)$ be fuzzy bounded linear operator on X where N be the fuzzy norm defined in eq.(2.1). Then

(1) $\rho(T)$ is nonempty set.

(2) $\rho(T)$ is τ-open where τ is the topology induced by the fuzzy norm defined by eq.(2.2).

Proof :

(1) By following the same first steps in proposition (2.6) one can get T is a bounded linear operator on $(X,\|.\|)$ then the resolvent set of T is nonempty [5], so there exists $\lambda\in\mathbb{C}$ such that $R_\lambda(T)$ exists, $R_\lambda(T)$ is bounded on $R(T_\lambda)$ and $R(T_\lambda)$ is dense in $(X,\|.\|)$. By [5] one can get $R_\lambda(T)$ is bounded on $(X,\|.\|)$. By following the same last steps in proposition (2.6) one can get $R_\lambda(T)$ exists and fuzzy bounded on X. Thus $\rho(T)$ is nonempty set.

(2) To show that $\rho(T)$ belongs to the topology τ induced by the fuzzy norm N defined by eq.(2.2). we shall prove that for all $c\in\mathbb{C}$, for each $t>0$ and for each α where $0<\alpha<1$, the set $B(c,\alpha,t)\in\tau$. Fix $c_1\in\mathbb{C}$, $t_1>0$ and $0<\alpha_1<1$. Let $c_2\in B(c_1,\alpha_1,t_1)$ this means that $N(c_1-c_2,t_1)>1-\alpha_1$. This implies $|c_1-c_2|<t_1$. To prove $B(c_2,\alpha_1,t_1-|c_1-c_2|)\subset B(c_1,\alpha_1,t_1)$, let $c_3\in B(c_2,\alpha_1,t_1-|c_1-c_2|)$ hence $N(c_2-c_3,t_1-|c_1-c_2|)>1-\alpha_1$ then $|c_2-c_3|<t_1-|c_1-c_2|$ and hence $|c_1-c_3|<t_1$ thus $N(c_1-c_3, t_1)=1>1-\alpha_1$ and hence $c_3\in B(c_1,\alpha_1,t_1)$. Therefore $B(c,\alpha,t)\in\tau$ for each $c\in\mathbb{C}$, $t>0$, $0<\alpha<1$. Moreover, we can see that $B(c,\alpha,t)=B(c,t)$ for each $\alpha\in(0,1)$, $t>0$ and $c\in\mathbb{C}$, where $B(c,t)=\{d\in\mathbb{C} \mid |c-d|<t\}$. Fix $c_1\in\mathbb{C}$, $0<\alpha_1<1$ and $t_1>0$. Let $c_2\in B(c_1,\alpha_1,t_1)$ then $N(c_1-c_2,t_1)>1-\alpha_1$. Hence $|c_1-c_2|<t_1$ thus $c_2\in B(c_1,t_1)$.

Conversely, suppose $c_2\in B(c_1,t_1)$ then $|c_1-c_2|<t_1$. Hence $N(c_1-c_2,t_1)>1-\alpha_1$. So $c_2\in B(c_1,\alpha_1,t_1)$. Therefore, $B(c,\alpha,t)=B(c,t)$, for each $\alpha\in(0,1)$, $c\in\mathbb{C}$, $t>0$. Then one can easily check the topologies induced by N and $|.|$ are the same. Then $\rho(T)$ is subset of resolvent set of T. Hence, $\rho(T)$ is τ-open.

Proposition (2.11):

Let $(X,\|.\|)$ be a complete normed space over the field $\mathbb{C}$, where $X\neq\{\mathbf{0}\}$ and let $T:(X,N) \longrightarrow (X,N)$ be fuzzy bounded linear operator on X where N be the fuzzy norm defined by eq.(2.2). Then

(1) $\rho(T)$ is nonempty set.

(2) $\rho(T)$ is τ-open where τ is the topology induced by the fuzzy norm defined by eq.(2.2).

Proof:

(1) By following the same first steps in proposition (2.7) one can get T is bounded linear operator on $(X,\|.\|)$ then the resolvent set of T is nonempty [5],

so there exists $\lambda\in\mathbb{C}$ such that $R_\lambda(T)$ exists, $R_\lambda(T)$ is bounded on $R(T_\lambda)$ and $R(T_\lambda)$ is dense in $(X,\|.\|)$. By [5] one can get $R_\lambda(T)$ is bounded on $(X,\|.\|)$. By following the same last steps in proposition (2.7) one can get $R_\lambda(T)$ exists and fuzzy bounded on X. Thus $\rho(T)$ is nonempty set.

The proof of (2) is similar to that in proposition (2.10).

Next, we give the definition of the spectrum of a linear operator on a fuzzy normed space over the field $\mathbb{C}$.

Definition (2.12):

Let (X,N) be a fuzzy normed space over the field $\mathbb{C}$ where $X\neq\{0\}$ and $T: X\longrightarrow X$ be a linear operator. The spectrum of T denoted by $\sigma(T)$ is the complement of $\rho(T)$ in the complex plane. Each $\lambda\in\sigma(T)$ is called a spectral value of T.

Remarks (2.13):

Let (X,N) be a fuzzy normed space over the field $\mathbb{C}$ where $X\neq\{0\}$ and $T:X\longrightarrow X$ be a linear operator the spectrum of T is partitioned into three disjoint sets as follows:-

(1) The point spectrum $\sigma_p(T)$ is the set such that $R_\lambda(T)$ does not exists. $\lambda\in\sigma_p(T)$ is called an eigenvalue of T.

(2) The continuous spectrum $\sigma_c(T)$ is the set such that $R_\lambda(T)$ exists and satisfying the condition (3) but not the condition (2) in definition (2.3).

(3) The residual spectrum $\sigma_r(T)$ is the set such that $R_\lambda(T)$ exists (and may be fuzzy bounded or not) but does not satisfy the condition (3) in definition (2.3).

To illustrate the definition of the pectrum of a linear operator defined on a fuzzy normed space over $\mathbb{C}$, consider the following examples.

Examples(2.14):

(1) Let (X,N) be a fuzzy normed space over the field $\mathbb{C}$ where $X\neq\{0\}$. Then $\sigma_p(I)=\{1\}=\sigma(I)$ where I is the identity operator defined on X. On the other hand, $\sigma_p(O) = \{0\} = \sigma(O)$, where O is the zero operator defined on X.

(2) Let $X=\mathbf{l}_2(\mathbb{C})$, that is

$$\mathbf{l}_2(\mathbb{C}) = \{x=(x_1,x_2,\mathbf{K})\ \sum_{i=1}^{\infty}|x_i|^2<\infty,\ x_i\in\mathbb{C}\}$$

For each $x\in\mathbf{l}_2(\mathbb{C})$, defined

$$\|x\|=<x,x>^{\frac{1}{2}}=\left(\sum_{i=1}^{\infty}|x_i|^2\right)^{\frac{1}{2}}.\text{Let}$$

$N:\mathbf{l}_2(\mathbb{C})\times\Re\longrightarrow[0,1]$ be the fuzzy norm defined by eq.(2.2). Consider $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$, where $x\in\mathbf{l}_2(\mathbb{C})$. We shall show that $0\in\sigma_r(T)$. To do this, it is clear T is bounded linear operator with respect to $\|.\|$. Moreover by using the same last steps in proposition (2.7) one can get T is fuzzy bounded on $\mathbf{l}_2(\mathbb{C})$. On the other hand,

$T: \mathbf{l}_2(\mathbb{C})\longrightarrow\mathbf{l}_2(\mathbb{C})$ is one to one. Then $T^{-1}:R(T)\longrightarrow\mathbf{l}_2(\mathbb{C})$ exists. Next, we show that

$R(T)=\{x\in\mathbf{l}_2(\mathbb{C})\mid x=(0,x_1,x_2,\ldots)\}$ is not dense in $\mathbf{l}_2(\mathbb{C})$. To do this, let $x\in\mathbf{l}_2(\mathbb{C})$ such that $x=(4,0,0,\ldots)$ and let $t=0.3>0$ and $\{x_n\}$ be any sequence in rang(T), that is, $x_n=(0, x_1^n, x_2^n, \ldots)$.Since $\|(0,x_1^n,x_2^n,\ldots)-(4,0,0,\ldots)\| = \|(-4, x_1^n, x_2^n,\ldots)$

$=(16+|x_1^n|^2+|x_2^n|^2+\ldots)^{1/2}$, for each n then $0.3<\|(-4, x_1^n, x_2^n, \ldots)\|$ for any choose for $x_1^n, x_2^n, \ldots$. Hence

$N(x_n-(4,0,0,\ldots),0.3)=$

$N((-4,x_1^n,x_2^n,\ldots),0.3)=0$ for each n. So $\lim_{n\to\infty}N(x_n-x,0.3)=0$. Thus for any sequence $\{x_n\}$ in $R(T)$ there exists $t=0.3>0$ such that $\lim_{n\to\infty}N(x_n-x,0.3)=0$ so x not belong to the closure of $R(T)$. Hence $R(T)$ is not dense in $\mathbf{l}_2(\mathbb{C})$. Then $\lambda=0\in\sigma_r(T)$.

Next, we give the definition of eigenspace of linear operator in a fuzzy normed space.

Definition (2.15):

Let (X,N) be a fuzzy normed space over the field $\mathbb{C}$ where $X\neq\{\mathbf{0}\}$ and $T:X\longrightarrow X$ be a linear operator and $\lambda\in\sigma_p(T)$. The subspace of X consisting of **0** and all eigenvectors of T corresponding to an eigenvalue λ of T is said to be the eigenspace of T corresponding to that eigenvalue λ.

The following propositions shows that $\sigma(T)=\sigma_p(T)$ and $\sigma(T)$ is τ-closed for a linear operator defined on a finite dimentional normed space.

Proposition (2.16):

Let $(X,\|.\|)$ be a finite dimensional normed space over the field $\mathbb{C}$ where $X\neq\{\mathbf{0}\}$ and $T:(X,N)\longrightarrow(X,N)$ be a linear operator where N be the fuzzy norm defined by eq.(2.1) then $\sigma(T)=\sigma_p(T)$ and $\sigma(T)$ is τ-closed where τ is the topology defined in remark (2.9) which is induced by the fuzzy norm defined by eq.(2.2).

Proof:

It is easy to check that the $\sigma(T)$ is nonempty. Suppose $\lambda\in\sigma(T)$ and $\lambda\notin\sigma_p(T)$ that is $T-\lambda I:(X,N)\longrightarrow(X,N)$ is one to one. On the other hand, by [5] and the same last steps in proposition (2.6) one can get $R_\lambda(T):(X,N)\longrightarrow(X,N)$ is fuzzy bounded on X. This means that $\lambda\in\rho(T)$ not $\lambda\in\sigma(T)$ and this is a contradiction. Hence $\lambda\in\sigma_p(T)$ and $\sigma(T)=\sigma_p(T)$. Moreover from proposition (2.10) one can prove $\sigma(T)$ is τ-closed.

Proposition (2.17):

Let $(X,\|.\|)$ be a finite dimensional normed space over the field $\mathbb{C}$ where $X\neq\{\mathbf{0}\}$, $T:(X,N)\longrightarrow(X,N)$ be a linear operator where N be the fuzzy norm defined by eq.(2.2) then $\sigma(T)=\sigma_p(T)$ and $\sigma(T)$ is τ-closed where τ is the topology defined in remark (2.9) which is induced by the fuzzy norm defined by eq.(2.2).

Proof:

It is easy to check that the spectrum of $T:(X,N)\longrightarrow(X,N)$ is nonempty. Suppose $\lambda\in\sigma(T)$ and $\lambda\notin\sigma_p(T)$ that is $T-\lambda I:(X,N)\longrightarrow(X,N)$ is one to one. On the other hand, $R_\lambda(T):(X,N)\longrightarrow(X,N)$ is fuzzy bounded on X. This means that $\lambda\in\rho(T)$ not $\lambda\in\sigma(T)$ and this is a contradiction. Hence $\lambda\in\sigma_p(T)$ and $\sigma(T)=\sigma_p(T)$. Moreover, $\sigma(T)$ is τ-closed follows from proposition (2.11).

The following propositions show that $\sigma(T)$ nonempty and it is τ-closed with respect to fuzzy bounded linear operators defined on the certain fuzzy normed spaces.

Proposition (2.18):

Let $(X,\|.\|)$ be infinite dimensional complete normed space over the field $\mathbb{C}$, and let $T:(X,N)\longrightarrow(X,N)$ be fuzzy bounded linear operator on X where N be the fuzzy norm defined by eq(2.1). Then is $\sigma(T)$ nonempty and it is τ-closed where τ is the topology defined in remark (2.9) which is induced by the fuzzy norm defined by eq.(2.2).

Proof:

By following the same first steps in proposition (2.6) one can get T is bounded linear operator on $(X,\|.\|)$. Then the spectrum of T is nonempty [5]. So there exists λ belong to spectrum of T. Then we have three cases:-

If λ belong to the point spectrum of T, then $T-\lambda I:(X,\|.\|)\longrightarrow(X,\|.\|)$ is not one to one. Hence $\lambda\in\sigma_p(T)$.

If λ belong to the continuous spectrum of T then $T-\lambda I:(X,\|.\|)\longrightarrow(X,\|.\|)$ is one to one, $R_\lambda(T)$: $(R(T_\lambda),\|.\|)\longrightarrow(X,\|.\|)$ is not bounded and $R(T_\lambda)$ is dense in X. Then $T-\lambda I$: $(X,N)\longrightarrow(X,N)$ is one to one, by the proof of proposition (2.6), $R_\lambda(T):(R(T_\lambda),N)\longrightarrow(X,N)$ is not fuzzy bounded and by the proposition (2.1) one can prove, $R(T_\lambda)$ is dense in X. Hence $\lambda\in\sigma_c(T)$.

If λ belong to the residual spectrum of T then $T-\lambda I$: $(X,\|.\|)\longrightarrow(X,\|.\|)$ is one to one and $R(T_\lambda)$ is not dense in X.

Then $T-\lambda I$: $(X,N)\longrightarrow(X,N)$ is one to one and by proposition (2.1) one can get $R(T_\lambda)$ is not dense in X. Hence $\lambda\in\sigma_r(T)$. Therefore $\sigma(T)$ with respect to N is nonempty. Also $\sigma(T)$ is τ-closed.

Proposition (2.19):

Let $(X,\|.\|)$ be infinite dimensional complete normed space over the field $\mathbb{C}$ and let $T:(X,N)\longrightarrow(X,N)$ be fuzzy bounded linear operator on X, where N be the fuzzy norm defined by eq.(2.2) and let. Then $\sigma(T)$ is nonempty and it is τ-closed where τ is the topology defined in remark (2.9) which is induced by the fuzzy norm defined by eq.(2.2).

Proof:

By following the same first steps in proposition (2.7) one can get T is bounded linear operator on $(X,\|.\|)$. Then the spectrum of T is nonempty [5]. So there exists λ belong to spectrum of T. Then we have three cases:-

If λ belong to the point spectrum of T, then $T-\lambda I: (X,\|.\|)\longrightarrow(X,\|.\|)$ is not one to one. Hence $\lambda\in\sigma_p(T)$.

If λ belong to the continuous spectrum of T then

$T-\lambda I: (X,\|.\|)\longrightarrow(X,\|.\|)$ is one to one, $R_\lambda(T):(R(T_\lambda),\|.\|)\longrightarrow(X,\|.\|)$ is not bounded and $R(T_\lambda)$ is dense in X. Then $T-\lambda I:(X,N)\longrightarrow(X,N)$ is one to one, by the proof of proposition (2.7), $R_\lambda(T):(R(T_\lambda),N)\longrightarrow(X,N)$ is not fuzzy bounded and by the proposition(2.2) one can prove, $R(T_\lambda)$ is dense in X. Hence $\lambda\in\sigma_c(T)$. If λ belong to the residual spectrum of T then

$T-\lambda I: (X,\|.\|)\longrightarrow(X,\|.\|)$ is one to one and $R(T_\lambda)$ is not dense in X. Then $T-\lambda I:(X,N)\longrightarrow(X,N)$ is one to one and by proposition (2.2) one an get

$R(T_\lambda)$ is not dense in X. Hence $\lambda\in\sigma_r(T)$. Therefore $\sigma(T)$ is nonempty. Also $\sigma(T)$ is τ-closed.

Proposition (2.20):

Let (X,N) fuzzy normed space over the field $\mathbb{C}$ where $X\neq\{0\}$ satisfying the following conditions

(1) For each α, $0<\alpha<1$ and for each sequence$\{x_n\}$ in X satisfying the condition $\lim_{n\to\infty} N(x_{n+p}-x_n,t)\geq\alpha$ for each $t>0$, $p=1,2,\ldots$, implies there exists $x\in X$ such that for each $t>0$,
$\lim_{n\to\infty} N(x_n-x,t)>\alpha$.

(2) For each $t>0$, $N(x,t)>0$ implies $x=\mathbf{0}$.

(3) For $x\neq\mathbf{0}$, $N(x,\cdot)$ is a continuous function of $\Re$ and strictly increasing on the subset $\{t:0<N(x,t)<1\}$ of $\Re$.

Let $T:(X,N)\longrightarrow(X,N)$ is fuzzy bounded, Then $\sigma(T)$ is nonempty.

Proof:

If T=I or T=O then $\sigma(T)\neq\phi\neq\sigma(O)$ follows from examples(2.14)(1). Suppose $I\neq T\neq O$ and $\sigma(T)=\phi$. Then $\rho(T)=\mathbb{C}$ hence for each $\lambda\in\mathbb{C}$, $T-\lambda I:(X,N)\longrightarrow(X,N)$ is one to one and by proposition (2.8), $R_\lambda(T):(X,N)\longrightarrow(X,N)$ is fuzzy bounded Then

$(T-\lambda I)$: $(X,\|.\|_\alpha)\longrightarrow(X,\|.\|_\alpha)$ is one to one and $R_\lambda(T)$: $(X,\|.\|_\alpha)\longrightarrow(X,\|.\|_\alpha)$ is a bounded linear operator for each $\alpha\in(0,1)$. Then λ belong to resolvent set of bounded linear operator T defined on complete normed space $(X,\|.\|_\alpha)$, for each $\alpha\in(0,1)$. So the resolvent set of T with respect to $\|.\|_\alpha$ is $\mathbb{C}$, for each $\alpha\in(0,1)$. Hence the spectrum of T is empty with respect to $\|.\|_\alpha$, for each $\alpha\in(0,1)$ and this is contradiction. So $\sigma(T)u\neq\phi$.

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الخلاصة

في هذا البحث قدمنا بعض التعاريف التي تتعلق بنظرية الطيفية للمؤثر الخطي T المعرف على الفضاء المعياري الضبابي حيث برهنا ان الطيف $\sigma(T)$ والمجموعة المحللة $\rho(T)$ تكونان غير خاليتان للمؤثر الخطي المقيد ضبابياً والمعرف على بعض الفضاءات المعيارية الخاصة . وكذلك قمنا ببرهان ان $\rho(T)$ تكون مفتوحة τ - و $\sigma(T)$ تكون مغلقة- τ .