δ (M) - Supplemented Modules

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Abstract

Let R be an associative ring with identity and M a non-zero unitary R-module. We introduce the concept of δ(M)– Supplement Submodule that if A,B≤M and M=A+B then B is called δ(M) – supplement of A if A∩B≤ δ(B) We give some properties of this kind of module.

1. Introduction and Preliminaries

For an associative ring with identity and a right R module M , a submodule N of M is said to be small in M (N <<M) if whenever N+X=M,then X=M. let U be a submodule of an R- module M, A submodule V< M is called supplement of U if V is minimal element in the set of submodules L<M with U+L = M ,V is a supplement of U if and only if U+V= M and U∩V<<V.

An R – module M is called supplemented if every submodule of M has supplement in M [1].

Let M be a module the concept of δ -small submodules was introduced by Zhou in [2]. Let M be an R-module and N ≤ M, N is said to be δ-small ($N \ll_\delta M$) if N+X=M with $\frac{M}{X}$ singular then X=M. A submodule N of an R–module M is called δ -supplement of L if M =N+L and $N+L \ll_\delta N$, M is called δ–supplemented module if for each submodule A of M there exists a submodule B of M such that M=A+B and $A\cap B \ll_\delta B$ [3]. A module M is called δ - hollow if every proper submodule of M is δ – small [4]. A submodule N of M is called essential in M if for every non-zero submodules L≤ M we have N∩L ≠ 0 and we write $N \leq_e M$

The following lemma show the properties of δ – small submodules .

Lemma 1.1 [2] :

let M be an R- module

1- For submodules N,K, L of M with N≤K we have $N \ll_\delta M$ if and only if $K \ll_\delta M$ and $\frac{N}{K} \ll_\delta \frac{M}{K}$ and $N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.
2- If $K \ll_\delta M$ and f: M→N is a homomorphism then $f(K) \ll_\delta M \leq N$ in particular if $K \ll_\delta M \leq N$ then $K \ll_\delta N$.
3- Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M=M_1+M_2$ then $K_1+K_2 \ll_\delta M_1+M_2$ if and only if $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$

Let M be an R-module and N≤M let $\delta(M) = \cap\{N \leq M \mid \frac{M}{N} \in \rho\}$ where ρ is the class of singular simple modules [3]. The following lemma shows some properties of δ(M).

Lemma (1.2) [2] :

1- δ (M) =∑{L≤ $\frac{M}{L}$ is δ - small submodule of M} .
2- If f : M → N an R-homomorphism then f(δ (M)) ≤ δ (N) .
3- If every proper submodule of M contained in a maximal submodule the δ(M) is largest δ–small submodule of M
4- If $M=\oplus_{i\in I} M_i$ then $\delta(M) = \oplus_{i\in I} \delta(M_i)$.

The concepts of generalized supplemented module introduced in [5], let M be a module if A, B ≤ M and M=A+B then B is called generalized supplement of A in case A∩B≤Rad (B). M is called generalized supplemented module if each submodule A has a generalized supplement B [6]. In this paper we introduce the concept of δ(M)–supplemented module as a generalized supplemented module (GS-module) and some properties of this kind of modules was given.

2. δ (M) – Supplemented Modules

Let M be a module. If A, B ≤ M and M = A+B then B is called a generalized supplement of A in case A∩B ≤ Rad (B) [2].

M is called a generalized supplemented module or GS-module in case each submodule A has a generalized supplement B. In this section as a generalization of generalized supplement submodule, δ(M)-supplemented modules are introduced many properties of δ(M)–supplemented module are given .

Definition 2 .1:

Let M be a module, and let A, B be submodules of M, B is called δ(M)– supplement of A, if M=A+B and A∩B≤ δ (B).

M is called a δ(M)–supplemented module in case each submodule A has a δ(M)–supplemented B. hollow modules and δ-hollow modules are δ(M)–upplemented module

It clear that M is δ(M)–supplemented of δ(M) in M.

Clearly each GS-module is δ(M)–supplemented module but the converse is not true in general as we see in the next remark.

Remark 2.2:

It is easy to check that if R is a semisimple ring and M a nonzero right R-module then M is nonsingular and semisimple. for any nonzero N≤ M , N is direct summand of M and hence is not small in M. but every submodule of M is δ-small in M then M is δ-hollow and then M is δ(M)–supplemented module.

Proposition 2 .3:

let A, B be submodules of an R- module M, if B is δ(M) supplement sub-module of A then:

1- If W+B=M for some W ⊂ A then B is a δ(M)–supplement of W.
2- If K $\ll_{\delta}$ M then B is δ (M) – supplement of A + K.
3- For K $\ll_{\delta}$M then K∩B $\ll_{\delta}$ B and so δ(B)=B∩δ (M) .
4- For L⊂A, (B+L) / L is δ(M)– supplement of $\frac{A}{L}$ in $\frac{M}{L}$.

Proof:

1. Let M=A+B, since B is δ(M) - supplement of A then A∩B ≤ δ(B) and A∩ B $\ll_{\delta}$B . Let W ≤ A, W∩B ≤ A∩B ≤ δ(B) then W∩B ≤ δ(B) we have B is δ(M) - supplemented of W .
2. If K$\ll_{\delta}$M then for X≤ B with (A+K)+X=M, A+X=M. Since B is δ(M) - supplemented of A and M=A+B then X=B. We have B is δ(M)- supplemented of A+K.
3. Let K$\ll_{\delta}$ M and X≤ B with M=A+B=A+ (K∩B) +X=A+X Then M=A+X is there for X=B and since $\frac{M}{B}$ is singular then $\frac{B}{X}$ is singular. That means K∩B $\ll_{\delta}$B. this yields B∩δ (M)≤δ(B). Since (B)≤V∩ δ(M) always holds we get δ(B)= B∩ δ(M)
4. For L≤A, we have A∩ (B+L) =B+(A∩L) by (Modularity) $\frac{A}{L}\cap\frac{B+L}{L}$Since A∩B≤ δ(B) [B is δ(M)-supplemented of A that means if A∩B$\ll_{\delta}$B then $\frac{A\cap B+L}{L} \ll_{\delta} \frac{B+L}{L}$] .

It follow that $\frac{A\cap B+L}{L} \leq \delta\left(\frac{B+L}{L}\right)$ Then $\frac{A}{L}\cap\frac{B+L}{L}\leq\left(\frac{B+L}{L}\right)$ and $\frac{A}{L}+\frac{B+L}{L}=\frac{M}{L}$

Lemma 2.4 [7]:

Suppose that $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ and $M=M_1 \oplus M_2$ then $K_1 \oplus K_2 \leq_e M_1 \oplus M_2$ if and only if $K_1 \leq_e M_1$ and $K_2 \leq_e M_2$.

Proposition 2.5 :

Let M be $\delta(M)$- supplemented modules then :

1- If A submodule of M with A∩$\delta(M)$ = 0 then A is semisimple
2- M=A+B for some semi simple and some module B with $\delta(B) \leq_e$ B.

Proof:

1- Let B≤A. Since M is $\delta(M)$ - supplemented module then there exists C ≤M such that B+C =M and B∩C≤ δ(C) thus A=A∩M =A∩(B+C)=B+A∩C we have A=B+(A∩C), B∩C≤δ(C) and

B∩(A∩C)=B∩C≤A∩ δ(C)≤ A∩ δ(M)= 0 We have B∩(A∩C)=0 since A=B⊕(A∩C) then A is semisimple.

2- For δ(M), let A≤M such that A∩δ(M)=0 and A⊕ δ(M)$\leq_e$ M see[2,prop.1.3].since M is δ(M)–supplemented module then there exist B ≤ M Such that M =A+B, A∩B≤ δ(B), A∩B=A∩(A∩B)≤ A∩δ (B) ≤ A∩δ(M)=0Then A∩B=0 by (1) M=A⊕B, A is semisimple Since δ(M)= δ(A)+δ(B)=δ (B) and since A⊕ δ(M) $\leq_e$M=A⊕B and A$\leq_e$A and δ(M)$\leq_e$B by [Lemma 2.4] δ(B)$\leq_e$B .

Proposition 2.6 :

Let A, B be submodules of R. module M. and A is δ(M)–supplemented module if A+B has δ(M)– supplement submodule in M then B is δ(M)–supplemented submodule.

Proof:

Since A+B be δ(M)- supplemented module then there exist X≤M such that X+(A+B)=M and X∩(A+B)≤ δ (X) For (X+B)∩A, since A is δ(M)– supplement submodule then there exist Y≤ A such that (X+B)∩A+Y=A and (X+B)∩Y≤δ(Y) since X+B+Y=M that is Y is δ(M)- supplement of X+B in M. Next show X+Y is δ(M)- supplement of B in M, since (X+Y)+B=0, so it is to show that (X+Y)∩B≤δ(X+Y). Since Y+B≤A+B, X∩(Y+B)≤X∩(A+B)≤δ(M), thus (X+Y) ∩B≤X∩(Y+B)+Y∩(X+B)≤ δ(X)+δ(Y)≤δ(X+Y)

Corollary 2.7 :

Let M_1,M_2 be δ(M)–supplemented module such that M=M_1 +M_2 then M is δ (M)–supplemented module .

Proof:

Let U be submodule of M, since M=M_1+M_2+U trivially has δ(M)–supplemented in M. M_2+U has δ(M)–supplemented in by [Proposition. 2.6] thus U has δ(M)– supplemented in M by [proposition. 2.6] so is M is δ(M)– supplemented module .

Proposition 2.8 :

Every factor module of δ(M)–supplemented module is δ(M) –supplemented module.

Proof:

Let M be δ(M)– supplemented module and $\frac{M}{N}$ any factor module of M, for any submodule L ≤ M cautioning N. since M is δ(M) – supplemented module then there exist K≤ M such that L+K=M and L∩K≤ δ(K) .

$\frac{M}{N} = \frac{L}{N} + \frac{K+N}{N}$ and $\frac{L}{N} \cap \frac{N+K}{N} = \frac{L\cap(N+K)}{N} = \frac{N+(L\cap K)}{N} \leq \delta\left(\frac{N+K}{N}\right)$ that is $\frac{N+K}{N}$ is δ(M) – supplemented module of $\frac{L}{N}$ in $\frac{M}{N}$.

Proposition 2.9 :

If M is δ(M) – supplemented module then $\frac{M}{\delta(M)}$ is semisimple.

Proof :

Let N ≤ M contain δ(M), there exist δ(M)–supplement submodule K of N in M such that M = N + K.

Since $\frac{M}{\delta(M)} = \frac{N}{\delta(M)} \oplus \frac{K+\delta(M)}{\delta(M)}$ then every submodule of $\frac{M}{\delta(M)}$ is direct summand. we have $\frac{M}{\delta(M)}$ is semi -simple.

3- δ (M)- amply supplement Modules

M is called generalized amply supplemented modules or briefly GAS-module in case M=A+B implies that A has a generalized supplement $K \leq B$.

In this section as a generalization of $\delta(M)$-supplemented module we introduce δ(M)-amply supplemented Modules

Definition 3.1 :

M is called δ (M)- amply supplemented modules in case M=A+B implies that A has a δ (M)- supplement $K \leq B$.

Is clear every δ (M)- supplemented module is δ(M)- amply supplemented module.

Proposition 3.2:

Let M be δ (M)-amply supplemented module and K a direct summand of M then K is a δ (M)- amply supplemented module.

Proof:

Since K is a direct summand of M, there exists $L \leq M$ such that M=K⊕L

suppose that K=C+D, then M=D+(C⊕L) since M is a δ(M)-amply supplemented module, there exist P≤D such that M=P+(C⊕L) and p∩$(C$⊕L)≤ δ(P). Therefore K=K∩M= K∩(P+(C⊕L)) =P+C and P∩C=P∩(C⊕L)≤ δ(P), as required.

Proposition 3.3:

Let M be a module. If every submodule of M is a $\delta(M)$- supplemented module, then M is a $\delta(M)$-amply supplemented module.

Proof :

Let K, N≤M ancl M=M+L. By assumption, there is H≤L such that (L∩N) +H = L and (L∩N) ∩ H=N∩ H ≤ δ(H). thus L=H+(L∩N) ≤ H+N and hence M=N+L≤ N+H .therefor M=H+N as required.

Corollary 3.4:

Let R be any ring. Then the following statement are equivalent:

1. Every module is a $\delta(M)$-amply supplemented module
2. Every module is a $\delta(M)$-supplemented-module.

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الخلاصة

لتكن R حلقة، M مقاسات في هذا البحث نقدم تعريف المقاس المكمل من النوع δ(M) الجزئي حيث اذا كان A، B مقاسات جزئية من M و M = B + A فأن B يكون مقاس جزئي مكمل من النوع$\delta(M)$ اذا كان A ∩ B ≤ δ(M) وكذلك قمنا بأعطاء خواص هذا النوع من الموديلات.